Master Thesis - Revised Version

L^0-groups, extreme amenability, and algebraic topology

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Disclaimer: This document is a revised edition of my master's thesis and is not identical to the official version submitted to Prof. Dr. Schneider. It includes corrections to spelling, wording, and the semantic content, as well as expanded sections—particularly on topological groups.

I would like to thank all proofreaders that have helped me correcting the errors in this thesis. And special thanks to Prof. Schneider who has answered many of my questions concerning the material discussed in this thesis.

Contents

List of Symbols

$[k, n]$ . . . . The set $[n] \setminus [k]$ for natural numbers $n \geq k$

$[n]$ . . . . The set $\{0, 1, \ldots, n-1\}$ for a natural number n

$\chi(G)$ . . . . The chromatic number of the graph G, i.e., the smallest number of colors needed for a proper vertex coloring

$\mathrm{dom}(f)$ . . . . The domain of the map f or relation f

id_X . . . . The map $X \to X$, $x \mapsto x$ on a set X

$\mathrm{im}(f)$ . . . . The image of the map f or relation f

$\mathbb{P}$ . . . . The set of prime numbers

$\mathbb{S}^n$ . . . . The n-dimensional sphere for $n \in \mathbb{N}$.

$\mathcal{N}(G)$ . . . . The set of open neighborhoods of the identity e_G of a topological group G

$\mathcal{N}_X(x)$ . . . . The set of all open sets in a top. space X containing the element $x \in X$

$\mathcal{T}_X$ . . . . The topology of the top. space X

$\mathbb{1}_x$ . . . . The indicator map for $x \in X$ on a set X

$|X|$ . . . . The cardinality of the set X

$\mathbb{N}$ . . . . The set of natural numbers including 0

$\mathbb{N}^+$ . . . . The set of natural numbers excluding 0

$\mathfrak{P}(X)$ . . . . The set of all subsets of X

$\mathfrak{P}_{\leq n}(X)$ . . . . The set of all subsets of X with cardinality less than or equal to $n \in \mathbb{N}$

$\mathfrak{P}_{\mathrm{fin}}(X)$ . . . . The set of all finite subsets of X

$\mathbb{R}$ . . . . The set of real numbers

$\mathbb{R}_{>0}$ . . . . The set of real numbers greater than 0

$\underline{\mathrm{Ab}}$ . . . . The category of abelian groups with group homomorphisms as arrows

$\underline{\mathrm{Ab}}(G, H)$ . . . . The class of all homomorphisms between the two abelian groups G and H

$\underline{\mathrm{Grp}}$ . . . . The category of groups with group homomorphisms as arrows

$\underline{\mathrm{Grp}}(G, H)$ . . . . The class of all homomorphisms between the two groups G and H

$\underline{\mathrm{Top}}$ . . . . The category of topological spaces with continuous functions as arrows

$\mathbb{Z}$	The set of integers
$\mathbb{Z}_p$	The group of integers modulo a prime p with addition modulo p
$B_\varepsilon(x)$	The open ball with radius $\varepsilon \in \mathbb{R}_{>0}$ in a metric space X around $x \in X$
$C(X, Y)$	The set of all continuous functions between top. spaces X and Y
d_X	The metric of the metric space X
$P(X, Y)$	Set of non-empty partial functions from X to Y
$X \sqcup Y$	Disjoint union of the sets X and Y, defined as, $X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$
X^c	The complement of the set X
X^Y	The set of all maps from Y to X

1 Introduction

The study of amenable groups is an important subfield of topological algebra. Amenability first appeared in the work of John von Neumann [9] in his investigation of the Banach-Tarski paradox. In von Neumann's formulation, a group is amenable if there exists a left-invariant mean on the group, meaning a finitely additive probability measure that remains unchanged under translations by group elements.

Later it was discovered that the amenability of a topological group is equivalent to a certain fixed-point property for continuous group actions. Specifically, a topological group is amenable if and only if every affine action of the group on a compact convex set admits a fixed point.

A stronger notion than amenability is extreme amenability. A topological group is said to be extremely amenable if every continuous action on a compact space has a fixed point, without additional restrictions on the nature of the action or the space.

In this thesis, we study a particular class of topological groups, denoted by $L_0(\mu, G)$. The elements of these groups are a special kind of limit of simple functions that take values in a topological group G and are measurable with respect to a submeasure μ. The structure of such groups depends on two main components: the submeasure μ and the topological group G.

These groups have been extensively studied under various conditions imposed on the submeasure μ and the group G. A central open question in this area concerns the extreme amenability of $L_0(\mu, G)$ when G itself is amenable.

In this thesis, we establish that $L_0(\mu, G)$ is extremely amenable when μ is a diffuse submeasure and G is an abelian topological group. Our approach links extreme amenability to a graph coloring problem. Specifically, we show that the group is extremely amenable if and only if a certain sequence of graphs admits no uniform finite bound on the number of colors required for a proper coloring. In Section 4, we introduce these graphs and formulate the coloring criterion. In Section 6, we prove a lower bound on the chromatic number of these graphs, using a monotonically increasing function related to their size. Section 5 provides an essential result concerning simplicial complexes associated with these graphs, which is necessary for the proof in Section 6.

Sections 2 and 3 develop the theoretical background required for the main arguments, covering fundamental topics from topology, algebraic topology, topological algebra, simplicial complexes, and homological algebra.

The general definitions of topology presented in Sections 2 and 3 can be found in any standard textbook on topology and algebraic topology. In Section 2, the definitions of general topology are primarily drawn from [7], with the concepts of filters and key results about them based on [11]. The definitions related to algebraic topology follow [8], while those concerning topological algebra are taken from [1].

Section 3 introduces simplicial complexes following [8], and the discussion on singular homology is inspired by Professor Martin Schneider's lecture Geometry and Topology, as well as [4]. Proofs of more advanced results are cited directly within the respective sections.

The key results in Sections 4, 5, and 6 are based on the work of Marcin Sabok [13].

All submeasures used in this thesis are submeasures defined on the power set algebra of a set X. This is possible because of the representation theorem of Stone from which follows that every boolean algebra can be represent as a power set algebra of a set.

2 Topology, Topological Algebra and Algebraic Topology

2.1 Topology

2.1.1 Topological Spaces

Definition 2.1:
Let X be a set. A **topology** $\mathcal{T}$ on X is a collection of subsets of X obeying the following axioms

1. $X \in \mathcal{T} \ni \emptyset$,
2. $\bigcup\limits_{\alpha \in I} A_\alpha \in \mathcal{T}$, where I is an arbitrary index set and $A_\alpha \in \mathcal{T}$ for all $\alpha \in I$,
3. $\bigcap\limits_{i=0}^{n} A_i \in \mathcal{T}$ for $n \in \mathbb{N}$ and $A_i \in \mathcal{T}$.

The tuple $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X, is called a **topological space**. ♠

In the following we will say that X is a topological space without mentioning the explicit topology every time.

Definition 2.2:
Let X be a topological space. A family of open sets $\mathcal{B} \subseteq \mathcal{T}_X$ is called a **basis** of the topology on X if

$$\forall U \in \mathcal{T}_X \forall x \in U \exists B \in \mathcal{B}\colon x \in B \subseteq U.$$

A family of open sets $\mathcal{S} \subseteq \mathcal{T}_X$ is called a **subbasis** of the topology on X if

$$\left\{ \bigcap \mathcal{S}' \colon \mathcal{S}' \in \mathfrak{P}_{\text{fin}}(\mathcal{S}) \right\}$$

is a basis of the topology on X. ♠

The intuitive idea of the definition above is that every open set in the topological space X can be represented by an arbitrary union of basis elements. Additionally each basis element is only a finite intersection of subbasis elements.

In many cases it is not possible to describe the topology on X directly and one can only give a basis for the topology. Then one can talk about $\mathcal{T}(\mathcal{B})$ which is the topology generated by the set $\mathcal{B}$. This topology consists of all arbitrary unions of elements in $\mathcal{B}$. The next lemma shows that not every subset of $\mathfrak{P}(X)$ is a basis for a well-defined topology topology on X.

Lemma 2.3:

Let X be a set and let $\mathcal{B} \subseteq \mathfrak{P}(X)$, then the following are equivalent:

- $\mathcal{B}$ is the basis of a topology on X,
- $\bigcup \mathcal{B} = X$ and $\forall B_1, B_2 \in \mathcal{B} \exists B_3 \in \mathcal{B} \colon \forall x \in B_1 \cap B_2 \colon x \in B_3 \subseteq B_1 \cap B_2$.

Proof. See [7, p. 78f.]. $\square$

Example 2.4:

Here are some examples of topological spaces which will be needed later.

i.) Let X be a nonempty set. Then the set $\mathcal{T}_X := \{X, \emptyset\}$ is a topology on X called the **chaotic topology**.

ii.) Let $(P, <)$ be a totally ordered set[1]. Define for $a, b \in P$ with $a < b$ the **open interval** from a to b as $(a, b) := \{x \in X \colon a < x \wedge x < b\}$ and similarly define the half open intervals $[a, b) := \{a\} \cup (a, b)$ and $(a, b] := (a, b) \cup \{b\}$. Then the **order topology** on P is generated by the basis

$$\mathcal{B} := \{(a, b) \colon a, b \in P\} \cup \{[a_0, b) \colon b \in P\} \cup \{(b, a_1] \colon b \in P\}$$

where a_0 is the smallest element in P if existent and a_1 is the largest element in P if existent.

Definition 2.5:

Let X and Y be topological spaces. A map $f \colon X \to Y$ is called **continuous** if for all $A \in \mathcal{T}_Y$ it follows that $f^{-1}(A) \in \mathcal{T}_X$. ♠

Definition 2.6:

Let X be a topological space and let $A \subseteq X$. A collection of subsets $(A_i)_{i \in I}$ of A where $A_i \in \mathfrak{P}(X)$ and I is an arbitrary index set is called a **cover** of A if it fulfills the following property

$$\bigcup_{i \in I} A_i = A. \tag{1}$$

[1] see [7, p. 24] for definition of total order (there simple order)

If the elements of the cover are open subsets of X the cover is called an **open** cover. A subcollection of a cover of A is called a **subcover** if it still fulfills Equation 1. ♠

Definition 2.7:

Let X be a topological space and $A \subseteq X$. A is called **compact** if every open cover of A has a finite subcover.

Furthermore X is called **locally-compact** if there exists a neighborhood basis of compact sets at each point $x \in X$. ♠

2.1.2 Filters

In a metric space (X, d_X), a sequence $(x_i)_{i\in\mathbb{N}}$ is said to converge to a limit point $x \in X$ if, for every $\varepsilon \in \mathbb{R}_{>0}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, the distance satisfies $d_X(x, x_n) < \varepsilon$. This definition of convergence, however, relies on specific topological properties of metric spaces and does not extend naturally to general topological spaces.

To address this limitation, topology introduces a more general concept of sequences known as **nets**. Nets extend the idea of sequences by allowing arbitrary index sets, making it possible to work with uncountable sequences. Despite this added flexibility, nets still share some of the same challenges as sequences—for example, selecting a convergent subsequence in a compact set.

Fortunately, an even more general notion of convergence exists in topology, which overcomes the limitations of both sequences and nets while still accommodating uncountable sequences. In the following this broader framework of convergence will be built up.

Definition 2.8:

Let X be a topological space, let $x \in X$ and let $(x_i)_{i\in\mathbb{N}}$ be a sequence in X. Then the sequence of the x_n converges to x ($x_n \to x$) if

$$\forall U \in \mathcal{N}_x \exists N \in \mathbb{N} \forall n \geq N : x_n \in U.$$

Let $A \subseteq X$ and $x \in X$. Then x is called an **accumulation point** of A if for every $U \in \mathcal{N}_x$ we can find $y \in U \cap A$ with $y \neq x$. ♠

This is a more general definition of convergence of sequences working in arbitrary topological spaces. The abstract definition of topological spaces makes this new notion of limits not as intuitive as in the special metric case. In some topological spaces there exist sequences that do not have a unique limit.

Example 2.9:

Consider the topological space of the real numbers $\mathbb{R}$ but with the chaotic topology (see 2.4 i.)). In this topological space, every sequence of numbers converges against every point. To this end consider a sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ and a real number x. We only have to check one open neighborhood of x namely the set $\mathbb{R}$. And this trivially fulfills the condition of Definition 2.8 because every sequence member is contained in it. So we have shown that an arbitrary sequence in this space converges against an arbitrary point.

This shows that additional assumptions have to be imposed onto a topological space such that the definition of convergence against a limit point makes sense.

Definition 2.10:

A topological space X is called **Hausdorff** if for any two points $x, y \in X$ there exists an open neighborhood U of x and an open neighborhood V of y which are disjoint. ♠

This property is enough such that every convergent sequence has a unique limit.

Theorem 2.11:

Let X be a topological space. If X is Hausdorff then every convergent sequence in X has a unique limit.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X and suppose it converges against x and y in X with $x \neq y$. Because of the assumption that X is Hausdorff there are open neighborhoods of the limit points U and V which are disjoint. Now we know that there are natural numbers n_0 and n_1 with the property

$$\forall n \geq n_0 \colon a_n \in U \ \wedge \ \forall n \geq n_1 \colon a_n \in V.$$

This means for $n := \max(n_0, n_1)$ that $a_n \in U \cap V = \emptyset$ ↯. □

The next example will show that this definition is not general enough for every topological space.

Example 2.12 ([11, 5.3 Beispiel])**:**
Let $(\Omega, \geq)$ be an uncountable set which is well-ordered, has a biggest element ω_1 and for all $\alpha \in \Omega$ with $\alpha < \omega_1$ the set $\{\beta \in \Omega \colon \beta \leq a\}$ is countable[2]. Now define the topological space Ω with the order topology like in 2.4 ii.) and $\Omega_0 := \Omega \setminus \{\omega_1\}$. It holds that ω_1 is an accumulation point of Ω_0 but there is no sequence in Ω_0 that converges to ω_1.

Proof. Suppose there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in Ω_0 such $\alpha_n \to \omega_1$. This means that $\sup_{n\in\mathbb{N}} a_n = \omega_1$. Now define

$$A_n = \{\beta \colon \beta \leq a_n\}$$

for all $n \in \mathbb{N}$. Since the sets A_n are all countable by definition of Ω the set

$$B := \bigcup_{n\in\mathbb{N}} A_n = \{\beta \in \Omega \colon \exists m \in \mathbb{N} \colon \beta \leq a_m\}$$

is also countable. This means the smallest element of $\Omega \setminus B$ is well-defined. Call it γ. Thus

$$\beta \in B \iff \beta \leq \gamma.$$

But by definition of Ω and the fact that $\gamma \in \Omega_0$ it follows that $\gamma < \omega_1$. Now we get

$$\sup_{n\in\mathbb{N}} a_n \leq \gamma < \omega_1. \ \lightning$$ □

This example illustrates the fact that not in every topological space, the standard definition of a sequence and convergence of a sequence is enough. There can be accumulation points that cannot be reached by any sequence. The problem is that the element ω_1 has an uncountable neighborhood basis (see 2.14) and sequences have only countable many elements such that the definition of convergence cannot be fulfilled by a sequence because of cardinality reasons. This problem cannot arise in metric spaces because the countability of all neighborhood basis is a condition for a topological space to be metrizable (see [7, p. 130f]).

In the following a more general definition of convergence in a topological space is developed.

[2]This construction is possible because of the well-ordering principle. See [11, p. 53]

Definition 2.13:

A **filter** $\mathcal{F}$ on a non-empty set X is a collection of subsets of X such that:

1. $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$,

2. $\forall A, B \in \mathcal{F}\colon A \cap B \in \mathcal{F}$,

3. $\forall A \subseteq B \subseteq X : A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$.

Denote by $\mathrm{Flt}(X)$ the set of all filters on X. Let $\mathcal{F}'$ be another filter on X. If $\mathcal{F} \subseteq \mathcal{F}'$ we call $\mathcal{F}'$ finer than $\mathcal{F}$ or $\mathcal{F}$ coarser than $\mathcal{F}'$. A family of subsets $\mathcal{F}_0 \subseteq \mathcal{F}$ of X is called a **filter basis** of $\mathcal{F}$ if

$$\forall F \in \mathcal{F} \exists F_0 \in \mathcal{F}_0\colon F_0 \subseteq F. \qquad \spadesuit$$

Definition 2.14:

Let X be a topological space. A **neighborhood basis** at $x \in X$ is a filter basis of the neighborhood filter $\mathcal{N}_X(x)$. ♠

Lemma 2.15:

Let $\mathcal{B}$ be a collection of non-empty subsets of a non-empty set X. $\mathcal{B}$ is a basis of a filter if and only if

$$\forall B_1, B_2 \in \mathcal{B} \exists B_3 \in \mathcal{B} : B_3 \subseteq B_1 \cap B_2. \tag{2}$$

Proof. Assume $\mathcal{B}$ is the basis of a filter $\mathcal{F}$ on X. It holds especially that $\mathcal{B} \subseteq \mathcal{F}$. Let $B_1, B_2 \in \mathcal{B}$. Since these are also filter elements it follows that $B_1 \cap B_2 \in \mathcal{B}$ which proves the claim.

Now assume that $\mathcal{B}$ fulfills equations 2 and let $B_1, B_2 \in \mathcal{B}$. Define the filter

$$\mathcal{F} := \{F \in \mathfrak{P}(X) \setminus \{\emptyset\}\colon \exists B \in \mathcal{B}\colon B \subseteq F\}.$$

It is clear that $\emptyset \notin \mathcal{F} \ni X$. Furthermore this filter is closed under the operation of taking supersets by definition. Now suppose that $F_1, F_2 \in \mathcal{F}$. There exist $B_1, B_2 \in \mathcal{B}$ with $B_1 \subseteq F_1$ and $B_2 \subseteq F_2$. Now by Equation 2 it follows that there is $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$ and thus

$$\emptyset \neq B_3 \subseteq B_1 \cap B_2 \subseteq F_1 \cap F_2.$$

Hence $F_1 \cap F_2 \in \mathcal{F}$. $\square$

From this point the notation $\mathcal{F}(\mathcal{B}) := \{F \in \mathfrak{P}(X) \colon \exists B \in \mathcal{B} \colon B \subseteq F\}$ represents the filter on X generated by the filter basis $\mathcal{B}$.

Corollary 2.16:

Let X be a non-empty set, $A \subseteq X$ and let $\mathcal{F}$ be a filter on X. If $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ then the set $\mathcal{A} := \{A \cap F \colon F \in \mathcal{F}\}$ forms a filter basis of a filter which is finer than $\mathcal{F}$.

Proof. Let $F_1, F_2 \in \mathcal{F}$ and define $A_1 := F_1 \cap A$ and $A_2 := F_2 \cap A$. Now consider

$$A_1 \cap A_2 = (F_1 \cap A) \cap (F_2 \cap A) = F_1 \cap F_2 \cap A = F_3 \cap A \in \mathcal{A}$$

where $F_3 = F_1 \cap F_2 \in \mathcal{F}$. Thus $\mathcal{A}$ is a filter basis by Lemma 2.15.

Now consider the filter $\mathcal{F}(\mathcal{A})$. Since $\mathcal{A}$ is a filter basis of this filter all elements of the form $F \cap A$ for $F \in \mathcal{F}$ are contained in $\mathcal{F}(\mathcal{A})$. For each $F \in \mathcal{F}$ there exists $F \cap A \in \mathcal{F}(\mathcal{A})$ and since $F \cap A \subseteq F$ it follows that $F \in \mathcal{F}(\mathcal{A})$. $\square$

Definition 2.17:

An **ultrafilter** $\mathcal{F}$ on X is a filter on X with the property that if there is another filter on X called $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ it follows that $\mathcal{F} = \mathcal{F}'$. Denote by $\mathrm{UFlt}(X)$ the set of all ultrafilters on X. If $\bigcap \mathcal{F} = \emptyset$ the ultrafilter is called **free**. ♠

Lemma 2.18:

Let X be a non-empty set and let $\mathcal{F} \in \mathrm{UFlt(X)}$ then

$$\forall A \subseteq X \colon A \in \mathcal{F} \vee A^c \in \mathcal{F}.$$

Proof. Since $A \cap A^c = \emptyset$ there are no two sets $F_1, F_2 \in \mathcal{F}$ with $F_1 \subseteq A$ and $F_2 \subseteq A^c$. This means that for all $F \in \mathcal{F}$ either $F \cap A \neq \emptyset$ or $F \cap A^c \neq \emptyset$. Assume w.l.o.g. that $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. From this it follows that $\{F \cap A \colon F \in \mathcal{F}\}$ is a filter basis of a filter $\mathcal{G}$ which is finer than $\mathcal{F}$ by Corollary 2.16. Since $\mathcal{F}$ is an ultrafilter it follows that $\mathcal{F} = \mathcal{G}$ and thus $A \in \mathcal{F}$. [11, 5.12 Satz] $\square$

Definition 2.19:

Let X be a topological space. A filter $\mathcal{F}$ converges to an element x in X ($\mathcal{F} \to x$) if $\mathcal{N}_x \subseteq \mathcal{F}$.

Let Y be another topological space and $f : X \to Y$ a continuous function. We call $f(\mathcal{F})$ the **image filter** of $\mathcal{F}$ under f which is the filter with filter basis $\{f(F)\colon F \in \mathcal{F}\}$. Another notation for the convergence of a filter is $\lim\limits_{F \to \mathcal{F}} f(F)$ (instead of $f(\mathcal{F}) \to y$). ♠

Indeed the new definition of convergence extends the old one from 2.8. Let X be a topological space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X which converges to $x \in X$. Then there exists a filter on X which converges to x namely $\mathcal{N}_X(x)$. The inverse is not true because the new definition of convergence now allows the convergence to the point ω_1 in Example 2.12. And it is also true that if $A \subseteq X$ and $a \in \bar{A}$ then there exists a filter that converges to a. This is not true for sequences by Example 2.12.

Theorem 2.20:

Let X be a topological space and let $A \subseteq X$. Then

$$x \in \bar{A} \iff \exists \mathcal{F} \in \mathrm{Flt}(X) : A \in \mathcal{F} \wedge \mathcal{F} \to x.$$

Proof. See [11, 5.17 Satz]. □

Definition 2.21:

The **Cofinite-Filter** $\mathcal{F}_{CF}$ on an infinite set X is defined as

$$\mathcal{F}_{CF} := \{F \in \mathcal{P}(X)\colon\ |F^c| < \infty\}.$$ ♠

Lemma 2.22:

Let X be an infinite set, then the Cofinite-Filter on X is a free filter.

Proof. It is trivial to see that $X \in \mathcal{F}_{CF}$ and $\emptyset \notin \mathcal{F}_{CF}$. So let $A, B \in \mathcal{F}_{CF}$ then

$$(A \cap B)^c = A^c \cup B^c \Rightarrow |(A \cap B)^c| = |A^c \cup B^c| \leq |A^c| + |B^c| < \infty$$

since A^c and B^c are finite. Now assume that $A \subseteq B$. It follows that

$$A \subseteq B \;\Rightarrow\; B^c \subseteq A^c \;\Rightarrow |B^c| \leq |A^c| < \infty.$$

This proves that $\mathcal{F}_{CF}$ is a filter on X. Now assume that $\bigcap \mathcal{F}_{CF}$ is non-empty and let $x \in \bigcap \mathcal{F}_{CF}$. Now let $F \in \mathcal{F}_{CF}$. By assumption it follows that $x \in F$ and

$$|(F \setminus \{x\})^c| = |F^c| + |\{x\}| = |F^c| + 1 < \infty$$

and hence $F \setminus \{x\} \in \mathcal{F}_{CF}$. $\lightning$ $\square$

Theorem 2.23:

Let X be non-empty set. Every filter on X is contained in an ultrafilter on X.

Proof. Let $\mathcal{F}$ be a filter on X and let Φ be the set of all filters on X that are finer than $\mathcal{F}$. This set forms a partial order together with the $\subseteq$ relation. If Φ_1 is a totally ordered subset of Φ then define $\mathcal{F}' := \bigcup_{\mathcal{G} \in \Phi} \mathcal{G}$. **Claim:** $\mathcal{F}'$ is a filter on X. The first property is clear because every filter contains X and does not contain $\emptyset$. Now let $A, B \in \mathcal{F}'$. This means that there are filters $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathcal{F}'$ with $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$. Since $\mathcal{G}_1 \in \Phi_1$ and $\mathcal{G}_2 \in \Phi_1$ it follows that $\mathcal{G}_1 \subseteq \mathcal{G}_2$ or $\mathcal{G}_2 \subseteq \mathcal{G}_1$. Suppose w.l.o.g. that $\mathcal{G}_1 \subseteq \mathcal{G}_2$. It follows that

$$A \in \mathcal{G}_2 \Rightarrow A \cup B \in \mathcal{G}_2 \Rightarrow A \cup B \in \mathcal{F}'.$$

Now suppose that $A \in \mathcal{F}'$ and $A \subseteq B \subseteq X$. Since there is an $\mathcal{G} \in \Phi_1$ with $A \in \mathcal{G}$ we can conclude that

$$B \in \mathcal{G} \Rightarrow B \in \mathcal{F}'.$$

This proves the claim. $\mathcal{F}'$ is an upper bound of Φ_1 and thus Φ_1 is inductively sorted. The existence of an ultrafilter on X follows from Zorn's Lemma. [11, 5.12 Satz] $\square$

Corollary 2.24:

Let X be a infinite set. There exists a free ultrafilter on X.

Proof. Follows directly from 2.22 and 2.23. $\square$

Lemma 2.25:

Every ultrafilter on a non-empty finite set X converges to a point of X.

Proof. In the case of a discrete set, the ultrafilter converges against an element if the singleton set containing this element is contained inside of the ultrafilter. Assume the cardinality of X is 1. Then the claim follows directly by the definition of the ultrafilter. Now assume that the Lemma is true for a set with cardinality $n \in \mathbb{N}$ and let X be a set with cardinality $n+1$ and additionally let $\mathcal{F}$ be an ultrafilter on X. Write $X = \{1, \dots, n\} \cup \{n+1\}$. By Theorem 2.18 either $\{n+1\} \in \mathcal{F}$ which ends the proof by the definition of convergence of an ultrafilter or $\{1, \dots, n\} \in \mathcal{F}$ which ends the proof by the inductive assumption. □

2.1.3 Uniform spaces

Definition 2.26:
Let X be a set and let $A, B \subseteq X \times X$ be realtions on X. Define the following other relations on X

- $A^{-1} := \{(a_2, a_1) \colon (a_1, a_2) \in A\}$,
- $A \circ B := \{(a, b) \colon \exists c \in X \colon (a, c) \in A \ \wedge \ (c, b) \in B\}$.

Additionally, define $A^2 := A \circ A$. ♠

So far, we have discussed a generalized concept of convergence in general topological spaces. In the analysis of topological groups, we also need the a generalized concept of completness of a space in terms of filters. This means a general notion of *cauchy sequences* is needed called *cauchy filters*. It will also be possible to say that a space is complete similiar to the metric case using this new notion.

Firstly, we will define a structure that arises naturally in the analysis of topological groups and is also compatible with filters and cauchy filters.

Definition 2.27:
Let X be a set. A non-empty subset $\mathcal{U} \subseteq \mathfrak{P}(X \times X)$ is called a **uniform structure** on X if

1. $\mathcal{U}$ is a filter,
2. $U \in \mathcal{U} \colon \{(x, x) \colon x \in X\} = \Delta_X \subseteq U$,
3. $U \in \mathcal{U} \colon U^{-1} \in \mathcal{U}$,
4. $U \in \mathcal{U} \exists V \in \mathcal{U} \colon V^2 \subseteq U$.

The elements of the uniform structure are called **entourages**. Let $E \in \mathcal{U}$ and let $x, y \in X$. The points x and y are called E-close if $(x, y) \in E$. Similiarly a subset $A \subseteq X$ is called E-small if $A \times A \subseteq E$.

Let X and Y be uniform spaces with uniform structures $\mathcal{U}_X$ and $\mathcal{U}_Y$. A map $f\colon X \to Y$ is called **uniformly continuous** if for each $W \in \mathcal{U}_Y$ there is $V \in \mathcal{U}_X$ such that

$$(f \times f)(V) \subseteq W$$

where

$$(f \times f)\colon X \times X \to Y \times Y, (x_1, x_2) \mapsto (f(x_1), f(x_2)).$$

A set $\mathcal{B} \subseteq X \times X$ is called **fundamental system of neighborhoods** of the uniform structure of X if every entourage E of X contains a set $B \in \mathcal{B}$. A non-empty fundamental system of neighborhoods is a filter basis for a uniform structure on X.

Let $A \subseteq X$ be a subset of X. Then the set

$$\mathcal{U}_A := \{E \cap (A \times A)\colon E \in \mathcal{U}_X\}$$

is a uniform structure on A induced by the uniform structure on X. ♠

The definition of uniform structures generalize metric spaces in a similiar sense that filter convergence generalizes convergence in metric spaces to a broader range of topological spaces. The axiom 2 of a uniform space reflects the fact that every point should be close to itself with respect to every entourage. This is similiar to the metric case, where the distance from a point to itself should be 0. In the case of uniform spaces, this condition is more general because we can have uniform spaces that are not Hausdorff. Axiom 3 of a uniform space is the equivalent to the axiom of symmetry a metric has to fulfill and axiom 4 is the equivalent to the triangle inequality.

A uniform space comes naturally equipped with a topology, that is induced by the uniformity of the space. This can easily be seen when considering a uniform space X with uniform structure $\mathcal{U}$. Then we can define a system of neighborhoods

$$\mathcal{N}_X(x) := \{V(x)\colon V \in \mathcal{U}\},\ V(x) := \{(x, y) \in X \times X\colon (x, y) \in V\}$$

for each point $x \in X$. The inverse is not true in general. But if the uniformity induces a topology, this topology is unique.

Theorem 2.28:

Let X be a uniform space with uniformity $\mathcal{U}$. Then this uniformity induces a unique topology on X.

Proof. See [11, Satz 11.5]. □

It is important to notice that there is not a one-to-one relation between uniform structures and the topologies induced by them. There can be multiple different uniformities that induce the same topology on the underlying space. This is especially true for topological groups which are equipped with three uniform structures that arise in a canonical way from the group topology and the group operation but all induce the same topology, namely the group topology.

Definition & Theorem 2.29:

Let X be a compact topological space. Then the set $\mathcal{E}(X)$ defined as

$$\mathcal{E}(X) := \mathcal{N}_{X\times X}(\Delta_X)$$

is a uniform structure on X which naturally arises from the topology on X.

Proof. See [2, p. 199f.]. □

Definition 2.30:

Let X be a uniform space with uniformity $\mathcal{U}$. A filter $\mathcal{F} \in \mathrm{Flt}(X)$ is called **cauchy** with respect to the uniform structure $\mathcal{U}$ if for each entourage $U \in \mathcal{U}$ there exists an element $F \in \mathcal{F}$ such that F is U-small, i.e., $F \times F \subseteq U$.

The uniform space X is called **Raikov-complete** (or **complete**) if every cauchy filter in X converges to a point in X. ♠

Example 2.31:

Let X be a topological space and let Y be uniform space with uniform structure $\mathcal{U}_Y$. Define for every $V \in \mathcal{U}_Y$ the following set

$$W(V) = \{(f,g) \in C(X,Y) \times C(X,Y) \colon \forall x \in X \colon (f(x), g(x)) \in V\}.$$

Then the set $\{W(V) \colon V \in \mathcal{U}_Y\}$ is a fundamental system of neighborhoods for the uniform structure of uniform convergence $C_u(X,Y)$ on the set $C(X,Y)$ of all continuous functions form X to Y. This space is Raikov-complete if Y is Raikov-complete.

Proof. See [11, p.183ff.]. □

Lemma 2.32:

Let X and Y be uniform spaces and let $f\colon X \to Y$ be a uniformly continuous map between them. If $\mathcal{F} \in \mathrm{Flt}(X)$ is cauchy in X then the image filter $f(\mathcal{F})$ is cauchy in Y.

Proof. Let E' be a entourage of Y and define $E := (f \times f)^{-1}(E')$ which is an entourage of X. Since $\mathcal{F}$ is cauchy in X there exists a filter element $F \in \mathcal{F}$ such that $F \times F \subseteq E$. But now it follows that

$$F \times F \subseteq E = (f \times f)^{-1}(E') \Rightarrow (f \times f)(F \times F) \subseteq (f \times f)(E) \subseteq E'.$$

We can conclude that $f(\mathcal{F})$ is cauchy in Y because $f(F) \in f(\mathcal{F})$. □

2.2 Topological Algebra

Definition 2.33:

Let G be a group with group operation $+_G$ and $\mathcal{T}$ a topology on the underlying set of G. Additionally define $\mathrm{mul}\colon G \times G \to G, (g,h) \mapsto g \cdot_G h$ as the multiplication map of G and $\mathrm{inv}\colon G \to G, g \mapsto g^{-1}$ as the inverse map of G. Then G is called a **topological group** if both mul and inv are continuous maps with respect to $\mathcal{T}$. Let X be a set. An action of G on X is a map $\lambda\colon G \times X \to X, (g,x) \mapsto \lambda(g,x) =: g\,.\,x$ that fulfills the following properties

1. $\forall g,h \in G : h\,.\,(g\,.\,x) = (h \cdot_G g)\,.\,x,$

2. $\lambda(e_G, x) = x$ for all $x \in X$.

If X is a topological space, then the action of G on X is continuous if λ is a continuous map. ♠

Example 2.34:

Let X be a compact, Hausdorff topological space. The set $\mathrm{Homeo}(X)$ of all homeomorphism from X to X together with the operation of composition of functions and taking inverses of functions is a group. It becomes a topological group when equipped with the compact-open topology.

Proof. Take $f, g \in \text{Homeo}(X)$. Since the composition of two continuous functions is continuous (see [11, Satz 2.20]) and composition of two bijections is again a bijection, $f \circ g$ is a continuous bijection. And since the inverses of f and g are continuous it is even a homeomorphism, which means that $f \circ g \in \text{Homeo}(X)$. Since $\text{id}_X \in \text{Homeo}(X)$ it follows that

$$\text{id}_X \circ f = f = f \circ \text{id}_X.$$

This means that id_X is the neutral element of the group. Lastely it is clear that $f \circ f^{-1} = \text{id}_X$ and $f^{-1} \circ f = \text{id}_X$, so every element has an inverse. Next we have to show that $\circ$ and taking function inverses are continuous maps. To this end assume that $W(C, W)$ is an open neighborhood of $g \circ f$. Firstly we now that $f(C) \subseteq g^{-1}(W)$ and that $f(C)$ is a compact set. We can choose for each $x \in f(C)$ a compact neighborhood V_x of x with $V_x \subseteq g^{-1}(W)$ since X is compact and thus locally-compact. Since $f(C)$ is compact there exists a finite set $F \subseteq f(C)$ such that

$$f(C) \subseteq \bigcup_{f \in F} \mathring{V}_f =: W'.$$

Also define the set $C' := \bigcup\limits_{f \in F} V_f$ which is compact.

So we can conclude that $f \in W(C, W')$ and $g \in W(C', W)$. And since $W' \subseteq C'$ it follows for all $f' \in W(C, W')$ and $g \in W(C', W)$ that

$$g' \circ f' \in W(C, W),$$

and hence $W(C, W') \circ W(C', W) \subseteq W(C, W)$. It follows that $\circ$ is continuous.

And it is also clear by definition that $f^{-1} \in \text{Homeo}(X)$ by definition of a homeomorphism. Let $W(K, U)$ be an open neighborhood of f^{-1} with $K \subseteq X$ compact and $U \subseteq X$ open. Define the closed set $A := U^c$ and the open set $V := K^c$. Since A is closed in X it follows that A is also compact in X. Also note that $K \cap f(A) = \emptyset$ since $K \subseteq f(U)$.

Now it follows that

$$f^{-1}(K) \subseteq U \iff K \subseteq f(U) \iff f(U)^c \subseteq K^c \iff f(A) \subseteq V,$$

hence $f \in W(A,V)$. Furthermore, for each $g \in W(A,V)$ it holds that

$$g(A) \subseteq V \iff V^c \subseteq g(A)^c \iff K \subseteq g(U) \iff g^{-1}(K) \subseteq U$$

and thus we get $g^{-1} \in W(K,U)$. We can conclude that $W(A,V)^{-1} \subseteq W(K,U)$ which means that taking inverses is continuous. □

Theorem 2.35:
Let X be a compact, Hausdorff topological space. The topological group Homeo(X) is Raikov-complete with respect to the two-sided uniformity $\mathfrak{S}$ of Homeo(X).

Proof. Let $\mathcal{F} \in \mathrm{Flt}(X)$ be a cauchy filter with respect to $\mathfrak{S}$. Since the inverse map is uniformly continuous, the filter $\mathcal{F}^{-1}$ defined as $\mathrm{inv}(\mathcal{F})$ is also a cauchy filter. Since the space $C(X,X)$ contains Homeo(X) and is complete (see 2.31) the filter limits $g := \lim \mathcal{F}$ and $h := \lim \mathcal{F}^{-1}$ exist in $C(X,X)$. Now consider

$$g \circ h = (\lim \mathcal{F}) \circ (\lim \mathcal{F}^{-1}) = \lim \mathcal{F} \circ \mathcal{F}^{-1}.$$

The exchange of function composition and the limit of filters is possible since the function composition is a jointly continuous map.
Let $E_0, E \in \mathcal{E}(X)$ such that

$$E^{-1} = E \quad \wedge \quad E \circ E \subseteq E_0.$$

Let $U := \{f \in \mathrm{Homeo}(X) \colon \forall x \in X : (g(x), f(x)) \in E\}$ which is a neighborhood of g in Homeo(X). By the definition of filter convergence we know that there exists $F \in \mathcal{F}$ such that $F \subseteq U$. Now let $(f, \tilde{f}) \in F \times F$ and observe for all $x \in X$ that

$$\Big(f(\tilde{f}^{-1}(x)), g(\tilde{f}^{-1}(x))\Big) \in E \wedge \Big(g(\tilde{f}^{-1}(x)), x\Big) \in E \Rightarrow \Big(f(\tilde{f}^{-1}(x)), x\Big) \in E_0.$$

The fact that $\Big(g(\tilde{f}^{-1}(x)), x\Big) \in E$ for all $x \in X$ follows by $\Big(g(x), \tilde{f}(x)\Big) \in E$ for all $x \in X$ and the substitution $y = \tilde{f}^{-1}(x)$. We can conclude that $F \circ F \subseteq \tilde{E}_0(\mathrm{id}_X)$.
From this is follows that $\lim \mathcal{F} \circ \mathcal{F}^{-1} = \mathrm{id}_X$ and thus $h = g^{-1}$, hence $g \in$ Homeo(X). □

In every topological group there arise three uniform structures in a natural way that all induce the group topology.

Definition 2.36:

Let G be a topological group. Define for each $V \in \mathcal{N}(G)$ the sets

$$R_V := \{(x,y) \in G \times G \colon xy^{-1} \in V\},$$
$$L_V := \{(x,y) \in G \times G \colon x^{-1}y \in V\}.$$

Now we can define the **right uniform structure** $\mathcal{R}$ and the **left uniform structure** $\mathcal{L}$ in the following way

$$\mathfrak{R} := \{E \subseteq G \times G \colon \exists V \in \mathcal{N}(G) \colon R_V \subseteq E\},$$
$$\mathfrak{L} := \{E \subseteq G \times G \colon \exists V \in \mathcal{N}(G) \colon L_V \subseteq E\}.$$

The two **sided uniform structure** $\mathfrak{S}$ on G can now be defined as

$$\mathfrak{S} := \mathfrak{R} \cap \mathfrak{L}.$$ ♠

2.2.1 Extreme Amenability

Definition 2.37:

A topological group G is called **extremely amenable** if every continuous action of G on a non-empty, compact, Hausdorff space admits a fixed point. ♠

Definition 2.38:

Let X and Y be sets. A step function $f \colon X \to Y$ with finite range induces a finite partition on X called $\mathcal{P}_f$ in the following way

$$\mathcal{P}_f := \{f^{-1}(\{y\}) \colon y \in Y\}.$$

This is a well-defined finite partion because of the assumption that f has finite range. ♠

Lemma 2.39:

Let G be a topological group, then it holds that

- $\forall U \in \mathcal{N}(G) \exists V \in \mathcal{N}(G) \colon V \cdot V \subseteq U$,
- $\forall U \in \mathcal{N}(G) \exists V \in \mathcal{N}(G) \colon V^{-1} \subseteq U$,
- $\forall U \in \mathcal{N}(G) \forall g \in G \exists V \in \mathcal{N}(G) \colon gVg^{-1} \subseteq U$.

Proof. See [11, Satz 16.16]. □

Definition 2.40:

Let X be a set and let $\mu\colon \mathcal{B} \to [0,\infty)$ be a submeasure on X on the boolean algebra $\mathcal{B} \subseteq \mathfrak{P}(X)$. Then the set

$$S(\mu, G) := \{f : X \to G\colon f \ \ \mu - \text{measurable} \wedge |f(X)| < \infty\}$$

for a topological group G is the set of all measurable step functions with finite range on X and values in G. ♠

Theorem 2.41:

Let G be a topological group and let $\mu\colon \mathfrak{P}(X) \supseteq \mathcal{B} \to [0,\infty)$ be a submeasure on X. Then the set of simple functions $S(\mu, G)$ together with the operation of pointwise multiplication

$$S(\mu, G) \times S(\mu, G) \to S(\mu, G),\ (f, g) \mapsto f \cdot g$$

where $(f \cdot g)(x) = f(x) \cdot g(x)$ for $x \in X$ and pointwise inverses

$$S(\mu, G) \to S(\mu, G),\ f \mapsto f^{-1}$$

where $(f^{-1})(x) = (f(x))^{-1}$ for $x \in X$ is a topological group with respect to the topology of convergence in submeasure. A basis element of this topology is given by

$$V_\varepsilon(f) = \{h \in S(\mu, G)\colon \mu(\{x \in X\colon h(x) \notin f(x)V\}) < \varepsilon\}$$

where $f \in S(\mu, G)$, $\varepsilon > 0$ and $V \in \mathcal{N}(G)$.

Proof. The fact, that $S(\mu, G)$ together with the pointwise multiplication of maps forms a group follows trivially from the fact that G is a group already. Hence, it remains to show that the maps mul and inv are continuous with respect to the group topology.

Let $\varepsilon \in \mathbb{R}_{>0}$ and let $V \in \mathcal{N}(G)$. Let $(g, h) \in S(\mu, G) \times S(\mu, G)$ and consider

$U := \mathrm{mul}^{-1}(V_\varepsilon(\mathrm{mul}(g,h)))$. Choose $\tilde{V} \in \mathcal{N}(G)$ such that $\tilde{V}^2 \subseteq V$. Next define

$$W := \left(\bigcap_{x \in h(X)} x^{-1} V_x x \right) \cap \tilde{V}$$

with $V_x \in \mathcal{N}(G)$ such that $x^{-1}V_x x \subseteq \tilde{V}$ for each $x \in h(X)$. Since the range of h is finite, W is an open set containing the identity of G. Let $(\tilde{g}, \tilde{h}) \in W_{\frac{\varepsilon}{2}}(g) \times W_{\frac{\varepsilon}{2}}(h)$. Now it follows that

$$\mu\left(\{x \in X \colon \tilde{g}(x) \notin g(x)W\}\right) < \frac{\varepsilon}{2} \wedge \mu\left(\left\{x \in X \colon \tilde{h}(x) \notin h(x)W\right\}\right) < \frac{\varepsilon}{2}$$

and

$$\begin{aligned}
&\mu\left(\left\{x \in X \colon \tilde{g}(x)\tilde{h}(x) \notin g(x)Wh(x)W\right\}\right) \\
&= \mu\left(\left\{x \in X \colon \tilde{g}(x)\tilde{h}(x) \notin g(x)h(x)(h(x))^{-1}Wh(x)W\right\}\right) \\
&\geq \mu\left(\left\{x \in X \colon \tilde{g}(x)\tilde{h}(x) \notin g(x)h(x)\tilde{V}W\right\}\right) \\
&\geq \mu\left(\left\{x \in X \colon \tilde{g}(x)\tilde{h}(x) \notin g(x)h(x)\tilde{V}^2\right\}\right) \\
&\geq \mu\left(\left\{x \in X \colon \tilde{g}(x)\tilde{h}(x) \notin g(x)h(x)V\right\}\right)
\end{aligned}$$

and since

$$\mu\left(\left\{x \in X \colon \tilde{g}(x)\tilde{h}(x) \notin g(x)Wh(x)W\right\}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

we get that

$$\mu\left(\left\{x \in X \colon \tilde{g}(x)\tilde{h}(x) \notin g(x)h(x)V\right\}\right) < \varepsilon.$$

Finally we can conclude that $\mathrm{mul}(W_{\frac{\varepsilon}{2}}(g) \times W_{\frac{\varepsilon}{2}}(h)) \subseteq U$.

Next consider the set Q defined as

$$Q = \bigcap_{y \in f(X)} yQ_y y^{-1}$$

with $Q_y \in \mathcal{N}(G)$ and $yQ_y y^{-1} \subseteq V$ for each $y \in f(X)$. Since the range of f is finite, Q is a open set containing the identity of G. It follows that

$$\begin{aligned}\tilde{f} \in \operatorname{inv}(Q_\varepsilon(f)) \iff \varepsilon &> \mu(\{x \in X \colon (\tilde{f}(x))^{-1} \notin f(x)Q\}) \\ &= \mu(\{x \in X \colon \tilde{f}(x) \notin Q^{-1}(f(x))^{-1}\}) \\ &= \mu(\{x \in X \colon \tilde{f}(x) \notin (f(x))^{-1}f(x)Q^{-1}(f(x))^{-1}\}) \\ &\geq \mu(\{x \in X \colon \tilde{f}(x) \notin (f(x))^{-1}V\})\end{aligned}$$

which means that $\tilde{f} \in V_\varepsilon(\operatorname{inv}(f))$ and thus $\operatorname{inv}(Q_\varepsilon(f)) \subseteq V_\varepsilon(\operatorname{inv}(f))$. □

Lemma 2.42:

Let H be a dense subgroup of a topological group G and let $f \colon H \to K$ be a homomorphism of H to a Raikov-complete topological group K, then there exists an extension of f to a continuous homomorphism $\hat{f} \colon G \to K$.

Proof. See [1, Proposition 3.6.12]. □

Theorem 2.43:

Let G be a topological group, let $H \subseteq G$ be a dense subgroup and let X be an arbitrary compact, Hausdorff topological space. Then every continuous action $H \curvearrowright X$ can be extended to a continuous action $G \curvearrowright X$.

Proof. Let $\lambda \colon H \times X \to X$ be a continuous action of H on X. Define the map $\varphi \colon H \to \operatorname{Homeo}(X)$ as

$$\varphi \colon H \to \operatorname{Homeo}(X),\ h \mapsto (x \mapsto \lambda(h, x))\,.$$

This map is well-defined since it is easy to show that every map $\lambda_h = \lambda(h, \cdot) \colon X \to X$ is a homeomorphism for each $h \in H$.

Claim: The map φ is a continuous homomorphism of topological groups. The fact that φ is a homomorphism of groups trivially follows from the fact of λ being a group action. It remains to be shown that the map is continuous. To this end let $K \subseteq X$ be non-empty and compact and let $U \subseteq X$ be non-empty and open and define $V := \lambda^{-1}(U) \subseteq H \times X$. Let $h \in \varphi^{-1}(W(K, U))$.

Firstly notice that

$$\forall x \in K \colon (h, x) \in V$$

since K is compact and since the map λ is continuous by assumption the set V is open in $H \times X$. This means we can choose $W_x \in \mathcal{N}_h(G)$ and $O_x \in \mathcal{T}_X$ such that $(h, x) \in W_x \times O_x \subseteq V$ for each $x \in K$. The sets O_x are an open covering of K. Hence, we can find a finite subcover $(O_{x_i})_{i \in \{1,\ldots,n\}}$ with $n \in \mathbb{N}$. Next define the set

$$W := \bigcap_{i=1}^{n} W_{x_i}$$

which is an open neighborhood of h in H and define

$$O := \bigcup_{i=1}^{n} O_{x_i}.$$

We know that by definition

$$\forall g \in W \forall i \in \{1, \ldots, n\} \colon \lambda(g, O_{x_i}) \subseteq U \Rightarrow \forall g \in W : \lambda(g, O) \subseteq U$$

and since $K \subseteq O$ it follows that

$$\forall g \in W \colon \lambda(g, K) \subseteq U$$

and thus $h \in W \subseteq \varphi^{-1}(W(K, U))$. Hence, the claim is proven.

Since H is a dense subgroup of the group G and by 2.35 the group $\mathrm{Homeo}(X)$ is Raikov-complete by 2.42 there exists a continuous extension of φ called $\hat{\varphi} \colon G \to \mathrm{Homeo}(X)$. The last step is to prove, that the map

$$\hat{\lambda} \colon G \times X \to X,\ (g, x) \mapsto \hat{\varphi}(g)(x)$$

is a continuous group action. The fact that this a group action follows trivially from the fact the $\hat{\varphi}$ is a homomorphism of groups.

It remains to show that the group action $\hat{\lambda}$ is continuous. To this end let $(g, x) \in G \times X$, choose $U \in \mathcal{T}_x$ with $\hat{\lambda}(g, x) \in U$.

Now consider $V := (\hat{\varphi}(g))^{-1}(U)$. This set is open since $\hat{\varphi}(g)$ is a homeomorphism and the set contains the point x since $\hat{\varphi}(g)(x) = \lambda(g, x) \in U$. The set $K := \bar{V}$ is compact because it is a closed subset of a compact space. Now take the basis element $W(K, U)$ of the compact open topology on $\mathrm{Homeo}(X)$ and define

$$W := \hat{\varphi}^{-1}(W(K, U)).$$

Again this set is open since $\hat{\varphi}$ is continuous and it contains g. Furthermore it is true that

$$\forall \tilde{g} \in W\colon \hat{\lambda}(\tilde{g}, K) = \hat{\varphi}(\tilde{g})(K) \subseteq U$$

and since $V \subseteq K$ thus

$$\forall \tilde{g} \in W\colon \hat{\lambda}(\tilde{g}, V) = \hat{\varphi}(\tilde{g})(V) \subseteq U.$$

We can now conclude that $\hat{\lambda}(W \times V) \subseteq U$ and since $(g, x) \in W \times V \in \mathcal{T}_{G \times X}$ the map $\hat{\lambda}$ is continuous. $\square$

2.3 Algebraic Topology

In the following section let Y be a topological space and let X be a metric space with metric d_X.

Definition 2.44:

Let X and Y be topological spaces. The continuous maps $f, g\colon X \to Y$ are called **homotopy equivalent** or **homotopic** if there exists a continuous map $F\colon X \times I \to Y$ where $I = [0, 1]$ with the following property

$$F(0, x) = f(x) \quad \text{and} \quad F(1, x) = g(x).$$

Define the relation $\simeq$ on $C(X, Y)$ as

$$f \simeq g\colon \iff f \text{ and } g \text{ are homotopy equivalent.}$$

This relation is an equivalence relation (see [7, Lemma 51.1]).

The continuous function f is called a **homotopy equivalence** if there exists a $h \in C(Y, X)$ such that

$$h \circ f \simeq \mathrm{id}_X \;\wedge\; f \circ h \simeq \mathrm{id}_Y.$$

The two spaces X and Y are **homotopy equivalent** ($X \simeq Y$) if there exists a homotopy equivalence $h \in C(X, Y)$ between them.

A space is called **contractible** if it is homotopy equivalent to a one point space.

♠

Lemma 2.45:

Let X and Y be topological spaces, then

$$X \cong Y \;\Rightarrow\; X \simeq Y.$$

Proof. Let $\psi\colon X \to Y$ be a homeomorphism. Then it follows that $\psi \circ \psi^{-1} \simeq \mathrm{id}_Y$ and $\psi^{-1} \circ \psi \simeq \mathrm{id}_X$. □

3 Simplicial complexes and singular homology

3.1 Abstract simplicial complexes

Definition 3.1:

Let X be a non-empty set. Then a collection $\mathcal{K} \subseteq \mathfrak{P}_{\text{fin}}(X)$ of finite subsets of X is called a **simplicial complex** of X if

- $\bigcup \mathcal{K} = X$,
- $\forall \sigma \in \mathcal{K} \colon \tau \subseteq \sigma \;\Rightarrow\; \tau \in \mathcal{K}$.

The elements of the set $V(\mathcal{K}) := \bigcup \mathcal{K}$ are the **vertices** of $\mathcal{K}$ and elements of $\mathcal{K}$ itself are the **simplices**. Given a simplex $\sigma \in \mathcal{K}$ then the sets $\sigma \setminus \{x\}$ for $x \in \sigma$ are called the **faces** of σ.

Let $\sigma \in \mathcal{K}$ then $\dim \sigma = |\sigma| - 1$ is the **dimension** of σ. The dimension of $\mathcal{K}$ is

$$\dim \mathcal{K} = \max\{\dim \sigma \colon \sigma \in \mathcal{K}\}.$$

Let Y be another non-empty set and $\mathcal{L}$ be a simplicial complex on Y. A map $f : V(\mathcal{K}) \to V(\mathcal{L})$ is called a **simplicial** map if

$$\forall \sigma \in \mathcal{K} \colon f(\sigma) \in \mathcal{L}.$$

If f is bijective and $f^{-1} \colon \mathcal{L} \to \mathcal{K}$ is a simplicial map then f is an **combinatorial isomorphism**. If $\iota \colon \mathcal{L} \to \mathcal{K}$ is an inclusion, we say that $\mathcal{L}$ is a **subcomplex** of $\mathcal{K}$ if ι is a simplicial map. ♠

Example 3.2:

Let $n \in \mathbb{N}^+$ then $\mathcal{K}_n$ defined as

$$\mathcal{K}_n := \mathfrak{P}_{<n}([n])$$

is a sequence of simplicial complexes. For example $\mathcal{K}_2 = \{\emptyset, \{0\}, \{1\}\}$.

Definition 3.3:

Let X be a set, H be a group and $\mathcal{K}$ be a simplicial complex on X. If there is a group action $\lambda \colon H \times X \to X$ of H on X such that $\lambda(h, \cdot) \colon X \to X$ is a simplicial map for each $h \in H$ then the complex $\mathcal{K}$ is an H-**complex**. ♠

Definition 3.4:

A poset (partially ordered set) is a pair $(P, \preceq)$ where P is a non-empty set and $\preceq$ is a binary relation on P which has the following properties:

1. *Reflexivity*: $\forall x \in P \colon x \preceq x$,
2. *Antisymmetry*: $\forall x, y \in P \colon (x \preceq y \ \wedge \ y \preceq x) \Rightarrow (x = y)$,
3. *Transitivity*: $\forall x, y, z \in P \colon (x \preceq y \ \wedge \ y \preceq z) \Rightarrow (x \preceq z)$.

A chain in P is a subset $C \subseteq P$ which is totally ordered, i.e.

$$\forall x, y \in C \colon x \preceq y \ \vee \ y \preceq x.$$ ♠

Example 3.5:

If X is a non-empty set then the pair $(\mathfrak{P}(X), \subseteq)$ forms a poset.

The last example establishes the following definition: If X is a non-empty family of sets then $P(X) := (X, \subseteq)$ is the poset generated by X. The elements of X are the elements of the poset and the binary relation of set inclusion is the partial order.

Definition 3.6:

Let $\mathcal{K}$ be an abstract simplicial complex. The **barycentric subdivision** $\mathrm{sd}(\mathcal{K})$ of $\mathcal{K}$ is the abstract simplicial complex with $\mathcal{K}$ as the set of vertices and all chains in $P(\mathcal{K})$ as simplices. ♠

Theorem 3.7:

If $\mathcal{K}$ is a abstract simplicial complex, then $\mathrm{sd}(\mathcal{K})$ is a well-defined abstract simplicial complex.

Proof. Since the singleton sets $\{\sigma\}$ for $\sigma \in \mathcal{K}$ are totally ordered we have that

$$\forall \sigma \in \mathcal{K} \colon \{\sigma\} \in \mathrm{sd}(\mathcal{K})$$

and thus $\bigcup \mathrm{sd}(\mathcal{K}) = \mathcal{K}$. Let $\sigma \in \mathrm{sd}(\mathcal{K})$. Now let $\tau \subseteq \sigma$. Since σ was totally ordered τ is also totally ordered. This means that $\tau \in \mathcal{K}$. □

We need a way to turn an abstract simplicial complex into a geometric structure such that it can be analyzed with the tools provided by algebraic topology. This means that there should be a map from the simplices of the abstract complex to n-dimensional euclidian space. This map is called the **geometric realization** of the abstract simplicial complex which maps it to a geometric simplicial complex. These geometric complexes consist of polyhedra. They are constructed by "gluing" together points, straight line segments, polyhedra and their higher dimensional generalizations. It will be clear that these two types of objects internally encode the same combinatorical structure.

Definition 3.8:

Let $n, k \in \mathbb{N}$ and let $x_0, x_1, \ldots, x_k \in \mathbb{R}^n$. The vectors are called **affinely independent** if

$$\forall \alpha_j \in \mathbb{R}\colon \sum_{i=0}^{k} \alpha_i = 0 \ \wedge \ \sum_{i=0}^{k} \alpha_i x_i = 0 \ \Rightarrow \ \alpha_0 = \alpha_1 = \ldots = \alpha_k = 0,$$

with $j = 0, 1, \ldots, k$. ♠

Definition 3.9:

Let $n, k \in \mathbb{N}$ and let $x_0, x_1, \ldots, x_k \in \mathbb{R}^n$. The convex hull of the vectors x_i is

$$\mathrm{co}(A) = \mathrm{co}(x_0, x_1, \ldots, x_k) := \{t_0 x_0 + t_1 x_1 + \cdots + t_k x_k \colon t_i \in [0,1], \sum_{i=0}^{k} t_i = 1\}$$

where $A = \{x_i \colon i \in [k+1]\}$. ♠

Remark 3.10:

Let $d \in \mathbb{N}$ and $A, B \subseteq \mathbb{R}^d$. If $A \cup B$ affinely independent, then

$$(\mathrm{co}A) \cap (\mathrm{co}B) = \mathrm{co}(A \cap B).$$

Proof. The inclusion $\mathrm{co}(A \cap B) \subseteq \mathrm{co}A \cap \mathrm{co}B$ is trivial.

For the other inclusion take $x \in \mathrm{co}(A) \cap \mathrm{co}(B)$. This means that there exist to representations of x of the form

$$x = \sum_{i=1}^{m} \alpha_i a_i,$$
$$x = \sum_{j=1}^{n} \beta_j b_j,$$

for $m, n \in \mathbb{N}$, $a_i \in A$ for all $i = 1, \dots, m$ with $\sum_{i=1}^{m} \alpha_i = 1$ and $b_j \in B$ for all $j = 1, \dots, n$ with $\sum_{j=1}^{n} \beta_j = 1$. Now subtract both representations of x. It follows that

$$\sum_{i=1}^{m} \alpha_i a_i - \sum_{j=1}^{n} \beta_j b_j = 0 \wedge \sum_{i=1}^{m} \alpha_i - \sum_{j=1}^{n} \beta_j = 0.$$

Since $A \cup B$ are affinely indepent this means that either all coefficients are zero or there exist indices for which $a_i = b_j$. Since the sum of the coefficients a_i and b_j sum up to one the case that all coefficients are zero is not possible and thus we have that both convex combinations of x can only contain points which lie in $A \cap B$. So $x \in \operatorname{co}(A \cap B)$. $\square$

Affine independence of the vectors ensures that the convex hull of vectors is not degenerated. For example it is expected that the convex hull of three vectors in $\mathbb{R}^2$ is a triangle. But in the degenerate case that the points are all the same or the points lie in a straight line, this is not true. When it is assumed that the points are affinely independent, these cases are excluded.

Definition 3.11:
A **geometric realization** of an abstract simplicial complex $\mathcal{K}$ is a map

$$f \colon V(\mathcal{K}) \to \mathbb{R}^d$$

with $d \in \mathbb{N}$ such that

1. $\forall \sigma \in \mathcal{K} \colon f(\sigma)$ affinely independent,
2. $\forall \sigma_1, \sigma_2 \in \mathcal{K} \colon (\operatorname{co} f(\sigma_1)) \cap (\operatorname{co} f(\sigma_2)) = \operatorname{co} f(\sigma_1 \cap \sigma_2)$.

Given a geometric realization f for $\mathcal{K}$, then $\|\mathcal{K}\|_f = f(\mathcal{K})$. The symbol $\|\mathcal{K}\|$ refers the an arbitrary geometric realization of $\mathcal{K}$ in the smallest dimension d that is possible. Let $x \in \|\mathcal{K}\|_f$. There exist maps $\alpha_v \colon \|\mathcal{K}\|_f \to [0, 1]$ for each $v \in V(\mathcal{K})$ such that

$$x = \sum_{v \in V(\mathcal{K})} \alpha_v(x) \cdot f(v_i), \qquad \sum_{v \in V(\mathcal{K})} \alpha_v(x) = 1,$$

The $\alpha_v(x)$ are called the **barycentric coordinates** of x. ♠

To show that $\|\mathcal{K}\|$ for an arbitrary simplicial complex is well-defined, we need the following theorem.

Theorem 3.12:

Let $\mathcal{K}$ be a simplicial complex with $\dim \mathcal{K} = n \in \mathbb{N}$. Then there exists a geometric realization $f: V(\mathcal{K}) \to \mathbb{R}^{2d+1}$.

Proof. This is a direct consequence of [6, Lemma 5.1.1]. □

Lemma 3.13:

Let $\mathcal{K}$ be a simplicial complex. Then

$$\|\mathcal{K}\| \simeq \|\mathrm{sd}(\mathcal{K})\|.$$

Proof. See [8, p. 84]. □

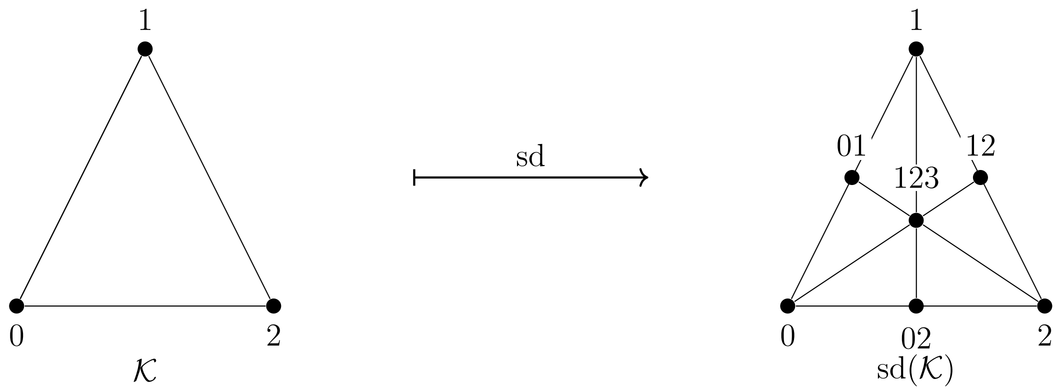

Figure 1: Barycentric subdivision of the 2-dimensional simplex.

Definition 3.14:

Let $d \in \mathbb{N}^d$ and let $f: V(\mathcal{K}) \to \mathbb{R}^d$. The **affine extension** $\|f\|$ of this function is defined as

$$\|f\|: \|\mathcal{K}\| \to \mathbb{R}^d,\ x \mapsto \sum_{v \in V(\mathcal{K})} \alpha_v(x) f(v).$$

♠

Example 3.15:

Take the sequence of simplicial complexes $\mathcal{K}_n$ from Example 3.2. A possible geometric realization of these complexes is

$$f_n: V(\mathcal{K}_n) \to \mathbb{R}^n, k \mapsto e_k$$

where e_k for $k \in [n]$ is the $k+1$-th canonical basis vector of $\mathbb{R}^n$. This sequence of complexes have the property that there is a homeomorphism $\psi: \|\mathcal{K}_n\|_{f_n} \to \mathbb{S}^{n-1}$.

Definition 3.16:

Let $\mathcal{K}$ and $\mathcal{L}$ be abstract simplicial complexes. The **join** of these complexes is defined as

$$\mathcal{K} \star \mathcal{L} := \{\sigma \sqcup \tau \colon \sigma \in \mathcal{K}, \tau \in \mathcal{L}\}.$$

The n-fold join $\bigstar_{i\in[n]}\mathcal{K}_i$ is defined by

$$\bigstar_{i=0}^{n-1}\mathcal{K}_i = \bigstar_{i\in[n]}K_i := \{\sigma_0 \sqcup \cdots \sqcup \sigma_{n-1} \colon \forall i \in [n] \colon \sigma_i \in \mathcal{K}_i\}.$$

Since the disjoint union is associative up to an isomorphism, this operation is well-defined. The special case $\mathrm{Cone}(\mathcal{K}) = \mathcal{K} \star \mathfrak{P}_{\leq 1}(\{\Delta\})$ is the cone of $\mathcal{K}$ with a cone point Δ. Given a finite subsequence of simplical subcomplexes $(L_i)_{i\in[n]}$ of $\mathcal{K}$ define

$$\mathrm{Cone}(\mathcal{K}, (L_i)_{i\in[n]}) := \mathcal{K} \cup \mathrm{Cone}(L_1) \cup \cdots \cup \mathrm{Cone}(L_{n-1})$$

for $n \in \mathbb{N}$. The cone points are distinct for each cone.

Now define basepoints $v_k \in V(\mathcal{K}), v_l \in V(\mathcal{L})$ of the two simplicial complexes. The **wedge** of $\mathcal{K}$ with $\mathcal{L}$ is defined as

$$\mathcal{K} \vee \mathcal{L} := {}^{\mathcal{K} \sqcup \mathcal{L}}\big/_{\sim}$$

where $\sim$ is the equivalence relation, that identifies the basepoints. The n-fold wedge $\bigvee_{i=0}^{n-1} \mathcal{K}_i = \bigvee\limits_{i\in[n]} \mathcal{K}_i$ for simplicial complexes $\mathcal{K}$ is the n-fold disjoint union with the same equivalence relation. ♠

Example 3.17:

Let $\mathcal{K} = \mathcal{L}$ with $V(\mathcal{K}) = [3]$ be the two-dimensional simplex (a triangle in the plain) and choose the basepoints $1 \in V(\mathcal{K}), 1 \in V(\mathcal{L})$. Then a geometric realization of the wedge $\mathcal{K} \vee \mathcal{L}$ is depicted in the Figure 2.

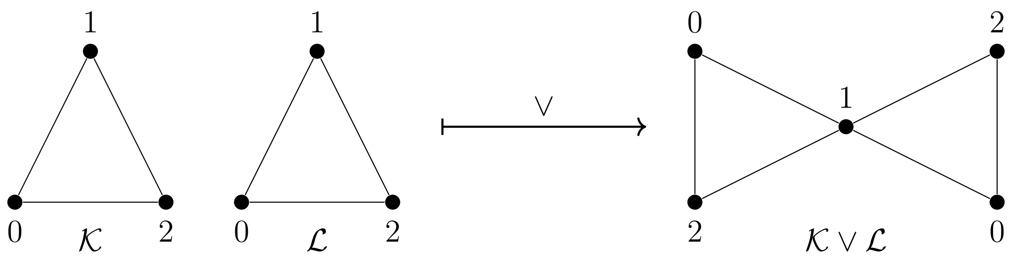

Figure 2: Examples of the wedge of simplicial complexes.

Lemma 3.18:

Let $\mathcal{K}$ be a simplicial complex and let $(L_i)_{i\in[p]}$ be a sequence of subcomplexes of $\mathcal{K}$ for $p \in \mathbb{N}$. If L_i is empty or contractible for each $i \in [p]$ then

$$\|\text{Cone}(\mathcal{K}, (L_i)_{i\in[p]})\| \simeq \|\mathcal{K}\|.$$

Proof. We know that each $\text{Cone}(L_i)$ for $i \in [p]$ is contractible. So there is a homotopy equivalence to a one point space. Choose a point in $\|\mathcal{K}\|$ to which $\text{Cone}(L_i)$ is contracted to for each L_i. Then we start with $i = 1$ and can define a homotopy equivalence that keeps $\|\mathcal{K}\|$ fixed and contracts the cone over L_i to the chosen point in $\|\mathcal{K}\|$. This shows that $\|\mathcal{K} \cup \text{Cone}(L_0)\| \simeq \|\mathcal{K}\|$. The statement follows by induction on i. □

Now consider the complexes $\mathcal{K} = \mathfrak{P}_{\leq 1}(\{0, 1\})$ and $\mathcal{L} = \mathfrak{P}_{\leq 1}(\{2, 3\})$. The geometric realization of the join $\mathcal{K} \star \mathcal{L}$ of these complexes can be seen in Figure 3. From the figure it becomese apperent that $\|\mathcal{K} \star \mathcal{L}\|$ is homeomorphic to $\mathbb{S}^1$.

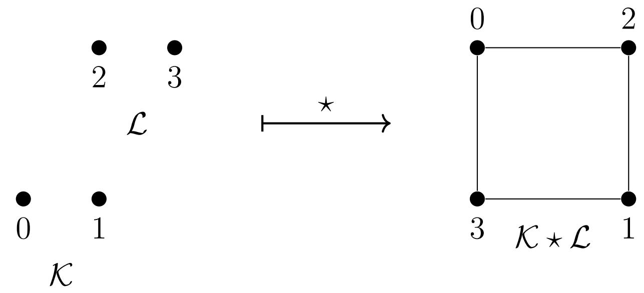

Figure 3: Example of the join of simplicial complexes.

The fact that the join of two points with two points results in a simplicial complex that has a geometric realization that is homeomorphic to $\mathbb{S}^1$ does not just work for dimension one. This fact holds more generally for the n-fold join of 2-point discrete simplicial complexes.

Lemma 3.19:

Let $\mathcal{K}_n := \mathfrak{P}_{\leq 1}(\{(-1, n), (1, n)\})$ for $n \in \mathbb{N}^+$, then

$$\left\| \bigstar_{i=1}^{k} \mathcal{K}_i \right\| \cong \mathbb{S}^{k-1}.$$

Proof. A geometric realization of this complex is as follows

$$f\colon V(\bigstar_{i=1}^{k}\mathcal{K}_i) \to \mathbb{R}^n,\ (x,i) \mapsto x \cdot e_i$$

which means that the vertices get mapped to the canonical basis vectors of $\mathbb{R}^n$ and their negations. This geometric realization forms the boundary of the n-crosspolytope[3] for each $n \in \mathbb{N}^+$. It is homeomorphic to the sphere by the homeomorphism

$$\varphi\colon \mathbb{S}^n \to \partial P^n,\ x \mapsto \|x\|_1^{-1} \cdot x. \qquad \square$$

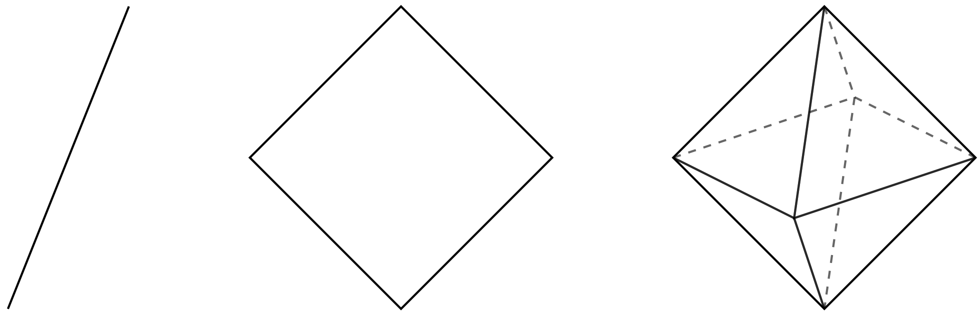

Figure 4: Cross polytopes in dimensions one, two and three.

3.2 Homology Theory

Now a very important tool in algebraic topology is introduced, namely **Homology Theory** of topological spaces. For the basic definitions from category theory and homological algebra needed in the following see [12].

Construction 3.20:

Let S be a finite set. Define

$$A(S) := \{f \in \mathbb{Z}^S \colon |\mathrm{im} f| < \infty\}$$

be maps from S to $\mathbb{Z}$ with finite image. $A(S)$ is an abelian group together with the operation of pointwise addition of maps. Then the set

$$\{\mathbb{1}_x \colon x \in S\} \subseteq A(S)$$

[3]This n-crosspolytope is $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon |x_1| + \cdots + |x_n| \leq 1\}$. The boundary ∂P^n of the polytope is formed by all points with 1-norm equal to 1.

forms a basis of the abelian group $A(S)$. Let T be another set. For a map $\varphi\colon S \to T$ define the map $A(T)$ as

$$A(\varphi)\colon A(S) \to A(T),\ f \mapsto \sum_{x\in S} f(x) \cdot \mathbb{1}_{\varphi(x)}.$$

This map is a homomorphisms of abelian groups.

Proof. The fact that $A(S)$ is abelian follows from the fact that the group $\mathbb{Z}$ is abelian and thus the pointwise addition is associative and commutative, the neutral element is the constant 0 function and the inverse of an element $f \in A(S)$ is defined as $f^{-1}(s) = -f(s)$ for $s \in S$.

Now let $f \in A(S)$ and consider the function

$$\tilde{f}\colon S \to \mathbb{Z},\ s \mapsto \sum_{x\in S} \alpha_x \mathbb{1}_x(s)$$

with $\alpha_x = f(x)$ for all $x \in S$. Let $s \in S$. It follows that

$$\tilde{f}(s) = \sum_{x\in S} \alpha_x \mathbb{1}_x(s) = \alpha_s = f(s).$$

Now let $f, g \in A(S)$.

$$\begin{aligned}
A(\varphi)(f + g^{-1}) &= \sum_{x\in S} (f + g^{-1})(x) \cdot \mathbb{1}_{\varphi(x)} \\
&= \sum_{x\in S} (f(x) - g(x))) \cdot \mathbb{1}_{\varphi(x)} \\
&= \sum_{x\in S} f(x) \cdot \mathbb{1}_{\varphi(x)} + (-g(x)) \cdot \mathbb{1}_{\varphi(x)} \\
&= \sum_{x\in S} f(x) \cdot \mathbb{1}_{\varphi(x)} + \sum_{x\in S} (-g(x)) \cdot \mathbb{1}_{\varphi(x)} \\
&= A(\varphi)(f) + A(\varphi)(g)^{-1}.
\end{aligned}$$

□

Since the collection of indicator functions is a basis of $A(S)$ it is a convention to write elements of $A(S)$ as formal linear combinations of elements of the generating set S. This means the group $A(S)$ can also be represented as

$$A(S) = \left\{ \sum_{i=1}^{k} r_i x_i \colon k \in \mathbb{N},\ \forall i \in \{1, \dots, k\} \colon x_i \in S, r_i \in \mathbb{Z} \right\}.$$

Theorem 3.21:

Let S be a set and let B be an abelian group. Then

$$\Phi \colon \underline{\mathrm{Ab}}(A(S), B) \to B^S,\ \varphi \mapsto \varphi \circ \iota^S$$

is a bijection where $\iota^S \colon S \to A(S),\ s \mapsto \mathbb{1}_s$.

Proof. The map Φ is a homomorphism of abelian groups.

To this end let $\varphi, \psi \in \underline{\mathrm{Ab}}$. The claim follows from the fact that φ and ψ are homomorphisms and

$$(\varphi + \psi)(\mathbb{1}_x) = \varphi(\mathbb{1}_x) + \psi(\mathbb{1}_x).$$

Let $\varphi \in \underline{\mathrm{Ab}}(A(S), B)$ and assume that $\varphi \circ \iota^S = 0$. Then it holds that

$$\begin{aligned}
\varphi(f) &= \varphi\left(\sum_{x \in S} f(x) \cdot \mathbb{1}_s\right) \\
&= \sum_{x \in S} f(x) \cdot \varphi(\mathbb{1}_s) \\
&= \sum_{x \in S} f(x) \cdot 0 = 0
\end{aligned}$$

for all $f \in A(S)$ and thus $\varphi = 0$. This means $\ker \Phi = \{0\}$. Now let $\psi \in B^S$. Then the map

$$\xi \colon A(S) \to B, f \mapsto \sum_{x \in S} f(x)\psi(x)$$

is a homomorphism by

$$\begin{aligned}
\xi(f + g) &= \xi\left(\sum_{x \in S} (f + g)(x) \mathbb{1}_x\right) \\
&= \sum_{x \in S} (f + g)\psi(x) \\
&= \sum_{x \in S} f(x)\psi(x) + \sum_{x \in S} g(x)\psi(x) = \xi(f) + \xi(g).
\end{aligned}$$

And now it follows that $\Phi(\xi) = (\xi \circ \iota^S)(x) = \mathbb{1}_x(x) \cdot \psi(x) = \psi(x)$ for all $x \in S$ and hence $\Phi(\xi) = \psi$. $\square$

The Theorem now tells us that any function from X to an abelian group B can be extended into a function from $A(X)$ to B in a unique way. This may become clear looking at the commutative diagram in Figure 5.

$$\begin{tikzcd} X \arrow[d, hook, "\iota^X"'] \arrow[dr, "\varphi"] & \\ A(X) \arrow[r, dashed, "\exists!\hat{\varphi}"'] & B \end{tikzcd}$$

Figure 5: Commutative diagram showing the statement of Theorem 3.21

.

Definition 3.22:
Let $n \in \mathbb{N}$. The **geometric n-simplex** Δ_n is defined as

$$\Delta_n := \mathrm{co}\{e_0, \ldots, e_n\} \subseteq \mathbb{R}^{n+1}$$

where e_i is the i-th canonical basis vector of $\mathbb{R}^{n+1}$ for $i = 0, \ldots, n$. For all $i = 0, \ldots, n$ define the i-th face map as

$$d_i \colon \Delta_{n-1} \to \Delta_n,\ (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

Notice that $\mathrm{im} d_i \subseteq \Delta_n$. The image of d_i is called the i-th face of Δ_n. ♠

Definition 3.23:
Let X be a topological space and let $n \in \mathbb{N}$. A **singular n-simplex** in X is a continuous map $\sigma \in C(\Delta_n, X)$. Now the **singular n-chain group** of X is defined as $C_n(X) := A(C(\Delta_n, X))$. An element $\sigma \in C_n(X)$ is called a singular n-chain in X. The map

$$\partial_n \colon C_n(X) \to C_{n-1}(X),\ \sigma \mapsto \sum_{i=0}^{n} (-1)^i (\sigma \circ d_i)$$

is called the n-th boundary map for $n \in \mathbb{N}^+$. It is a homomorphisms of abelian groups. For $n < 1$ define ∂_n as the zero map as well as all C_n as trivial abelian groups.

If Y is another topological space and $f \in C(X, Y)$ then $C_n(f) \colon C_n(X) \to C_n(Y)$

defined on the generators $\sigma \in C_n(X)$ as

$$C(f)(\sigma) = f \circ \sigma.$$

This map is a homomorphism of abelian groups. ♠

Lemma 3.24:

The composition $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}$.

Proof. See [14, Lemma 8.7]. □

The Lemma above is the so called Fundamental Theorem of Homology Theory. It allows for the following definition.

Definition 3.25:

Let X be a topological space, then

$$C_\bullet(X) := ((C_n(X))_{n\in\mathbb{Z}}), (\partial_n)_{n\in\mathbb{Z}})$$

with

$$C_n(X) = \{0\},\ \partial_n := 0$$

for $n \in \mathbb{Z} \setminus \mathbb{N}$ is a chain complex. ♠

Lemma 3.26:

Let X, Y be topological spaces and $f \in C(X, Y)$, then

$$\forall n \in \mathbb{N}\colon \partial_n \circ C_n(f) = C_{n-1}(f) \circ \partial_n.$$

Proof. Let $n \in \mathbb{N}^+$ and let $\sigma \in C(\Delta_n, X)$ then

$$\begin{aligned}
\partial_n(C_n(f)(\sigma)) &= \partial_n(f \circ \sigma) \\
&= \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ d_i \\
&= \sum_{i=0}^{n} (-1)^i C_{n-1}(f)(\sigma \circ d_i) \\
&= C_{n-1}(f)(\sum_{i=0}^{n} (-1)^i (\sigma \circ d_i)) \\
&= C_{n-1}(f)(\partial_n(\sigma)).
\end{aligned}$$

□

From the previous lemma it is clear that $C_\bullet(f) := (C_n(f))_{n\in\mathbb{Z}}$ with $C_n(f) = 0$ for $n < 0$ is a homomorphism between chain complexes $C_\bullet(X)$ and $C_\bullet(Y)$.

Lemma 3.27:

$C_\bullet\colon \underline{\mathrm{Top}} \to \underline{\mathrm{C}}(\underline{\mathrm{Ab}})$ is a functor.

Proof. Let X be a topological space and let $n \in \mathbb{N}$, then

$$C_n(\mathrm{id}_X)(\sigma) = \mathrm{id}_X \circ \sigma = \sigma = \mathrm{id}_{C_n(X)}(\sigma)$$

for all $\sigma \in C(\Delta_n, X)$ and hence $C_n(\mathrm{id}_X) = \mathrm{id}_{C_n(X)}$. For $n < 0$ this is trivial.

Let Y, Z be topological spaces and $f \in C(X,Y), g \in C(Y,Z)$, then for $n \in \mathbb{N}$ it follows that

$$C_n(g \circ f)(\sigma) = g \circ f \circ \sigma = C_n(g)(f \circ \sigma) = (C_n(g) \circ C_n(f))(\sigma)$$

for all $\sigma \in C(\Delta_n, X)$ and thus $C_n(g \circ f) = C_n(g) \circ C_n(f)$. The case for $n < 0$ is trivial. □

Theorem 3.28:

Let $n \in \mathbb{N}$. Then

$$H_n\colon \underline{\mathrm{Top}} \to \underline{\mathrm{Ab}}$$

defined by

$$H_n(X) := H_n(C_\bullet(X)),\ H_n(f) := H_n(C_\bullet(f))$$

for a topological spaces X and Y and $f \in C(X,Y)$ is called the n-th singular homology group and is a functor. The definition of this functor is part of homological algebra and can be found in [12, p. 329f.].

Proof. Follows from the fact that the composition of functors is a functor and Lemma 3.27. □

Theorem 3.29:

Let X and Y be topological spaces. Then if $f, g \in C(X,Y)$ are homotopic maps it follows that

$$H_n(f) = H_n(g)$$

for all $n \in \mathbb{N}$.

Proof. See [4, p. 112f]. □

Corollary 3.30:
Let X and Y be topological spaces and $f \in C(X,Y)$ is a homotopy equivalence between them, then $H_n(f)\colon H_n(X) \to H_n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$.

Proof. This follows directly from Theorem 3.29. □

Example 3.31:
Take a topological space X. Then $H_0(X) \cong \mathbb{Z}^n$ where $n \in \mathbb{N}$ is the number of path-connected components in X.

In classical homology theory, the 0-th homology group of a space counts its connected components, which can sometimes lead to complications in certain results. For example, the homology of a single-point space is typically $H_0 = \mathbb{Z}$, but this does not always align well with other homological computations.

To address this, we introduce **reduced homology groups**, denoted $\tilde{H}_n(X)$. These groups are defined such that:

$$\tilde{H}_n(X) = H_n(X) \quad \text{for all } n \geq 1, \quad \text{and} \quad \tilde{H}_0(X) \oplus \mathbb{Z} = H_0(X).$$

This adjustment ensures that the reduced homology of a single-point space is trivial, $\tilde{H}_0 = 0$, rather than $\mathbb{Z}$, simplifying various theoretical results. A detailed derivation of reduced homology can be found in [4, p. 110].

Beyond using integer coefficients, homology can also be defined with coefficients in other groups, such as $\mathbb{Z}_p$ for a prime p. The fundamental ideas remain the same, with slight modifications in proofs and derivations. Since this thesis focuses on homology over $\mathbb{Z}_p$, we refer to [4, p. 153ff.] for a comprehensive discussion of homology with different coefficients.

Definition 3.32:
Let G_α be an abelian group for each α in an arbitrary index set I. Then the **external direct sum** of the groups is defined as

$$\bigoplus_{\alpha \in I} G_\alpha = \{(g_\alpha)_{\alpha \in I}\colon g_\alpha \in G_\alpha \ \wedge \ g_\alpha \neq e_{G_\alpha} \text{ for only finitely many indices } \alpha \in I\}.$$

This is an abelian group with the operation

$$(g)_{\alpha \in I} \cdot (h)_{\alpha \in I} = (g_\alpha \cdot_{G_\alpha} h_\alpha)_{\alpha \in I}.$$

If f_α is a homomorphism with

$$f_\alpha \colon G_\alpha \to H$$

for each $\alpha \in I$ where H is an abelian group then

$$\bigoplus_{\alpha \in I} f_\alpha \colon \bigoplus_{\alpha \in I} G_\alpha \to H,\ (g)_{\alpha \in I} \mapsto \sum_{\alpha \in I} f_\alpha(g_\alpha)$$

is a well-defined homomorphism since only finitely many summands are non-trivial.

♠

Definition 3.33:

Let X be a topolgical space and let $A \subseteq X$ be a non-empty closed subspace, that is a deformation retract of a neighborhood in X. Then (X, A) is a **good pair**. ♠

Lemma 3.34:

Let X_α be a sequence of topological spaces with basepoints $x_\alpha \in X_\alpha$ and let $\iota_\alpha \colon X_\alpha \to \bigvee_{\alpha \in I} X_\alpha$ for $\alpha \in I$ where I is an arbitrary index set such that (X_α, x_α) are good. Then it holds that the maps ι_α induce an isomorphism

$$\bigoplus_{\alpha \in I} H_n(\iota_\alpha) \colon \bigoplus_{\alpha \in I} \tilde{H}_n(X_\alpha) \to \tilde{H}_n\left(\bigvee_{\alpha \in I} X_\alpha\right).$$

Proof. See [4, Corollary 2.25]. □

Proofs in the later sections will focus solely on reasoning about homology groups. While there exists a dualized concept known as **cohomology**, where the n-th cohomology group is denoted by $H^n(X; G)$, where G is a module over a field, its development is not necessary for this thesis.

In categorical terms, cohomology arises from a dualization of homology, which replaces the homology groups in the chain complex with $\underline{\mathrm{Grp}}(H_n(X), G)$, where $G \in \underline{\mathrm{Ab}}$ is an abelian group. However, there is no need to introduce cohomology explicitly or utilize it in later proofs due to the following fundamental result:

Alexander Duality, which establishes a connection between homology and cohomology groups.

Theorem 3.35:
Let $n \in \mathbb{N}$. If $A \subseteq \mathbb{S}^n$ is a compact and non-empty subspace and $(\mathbb{S}^n, A)$ is triangulable[4] then

$$\tilde{H}^k(A) \cong \tilde{H}_{n-k-1}(\mathbb{S}^n \setminus A)$$

for all $k \in \mathbb{N}$.

Proof. See [8, Theorem 71.1]. □

Definition 3.36:
Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of abelian groups and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of homomorphisms such that

$$\varphi_n \colon G_n \to G_{n-1}$$

for $n \in \mathbb{N}^+$ and $\varphi_0 \colon G_0 \to 0$. The sequence is called **exact** if

$$\mathrm{im}\varphi_{n+1} = \ker \varphi_n.$$

for all $n \in \mathbb{N}$. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a **short exact sequence**. ♠

A second important tool linking homology and cohomology groups is the **Universal Coefficient Theorem**. An in-depth derivation of it can be found in [4, Chapter 3.1]. For the purpose of this thesis, we only need the following corollary from the theorem.

Corollary 3.37:
Let X be a topological space. Let $C_\bullet(X; R)$ be a chain complex of free abelian groups and let R be a field and $H_n(X; R)$ the corresponding homology groups with $\tilde{H}_n(X; R) \cong 0$ for $0 \le n \le k \in \mathbb{N}$. Then the cohomology groups are all trivial up to k.

[4] This means that $A \subseteq \mathbb{S}^n$ and that there exists a triangulation $\mathcal{K}$ of $\mathbb{S}^n$ and a subcomplex $\mathcal{K}_0 \subseteq \mathcal{K}$ with a homeomorphism $(\mathcal{K}, \mathcal{K}_0) \to (\mathbb{S}^n, A)$.

We will use this Corollary in the case $R = G = \mathbb{Z}_p$. This works since $\mathbb{Z}_p$ is a field and a free $\mathbb{Z}_p$-module over itself in the case $p \in \mathbb{P}$.

3.3 Theorem of Mayer-Vietoris

Often we can calculate the homology of simple subspaces of a larger space and want to deduce the homology of the larger space from them. This can be accomplished with the **Mayer-Vietoris-Sequence** which is a homological version of the **Van Kampen-Theorem** (see [4, Chapter 1.2]).

Theorem 3.38:
Let X be a topological space and let $A, B \subseteq X$ subspaces of X such that

$$X = A \cup B,\ A \cap B \neq \emptyset.$$

The **Mayer-Vietoris-Sequence**

$$\ldots \to H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(A \cup B) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \ldots \to 0$$

is an exact sequence where the mappings Φ and Ψ are defined as

$$\Phi\colon H_n(A \cap B) \to H_n(A) \oplus H_n(B),\ z + B_n(A \cap B) \mapsto (z + B_n(A), -z + B_n(B)),$$
$$\Psi\colon H_n(A) \oplus H_n(B) \to H_n(A \cup B),\ (z_1 + B_n(A), z_2 + B_n(B)) \mapsto (z_1 + z_2) + B_n(A \cup B).$$

Proof. For a detailed derivation of this Theorem look at [4, p. 149]. □

Example 3.39:
The homology groups of the n-dimensional sphere $\mathbb{S}^n$ are as follows

$$\tilde{H}_k(\mathbb{S}^n) \cong \begin{cases} 0, & n \neq k, \\ \mathbb{Z}, & n = k, \end{cases}$$

for $k \in \mathbb{Z}$.

Proof. Let A be the upper hemisphere and B the lower hemisphere of $\mathbb{S}^n$. Then

$A \cap B = \mathbb{S}^{n-1}$. Consider the Mayer-Vietoris sequence of this decomposition

$$\ldots \xrightarrow{\partial'} H_k(\mathbb{S}^{n-1}) \to H_k(A) \oplus H_k(B) \to H_k(\mathbb{S}^n) \xrightarrow{\partial} H_{k-1}(\mathbb{S}^{n-1}) \to \ldots \to 0.$$

Since A and B are contractible it follows that $H_n(A) \cong H_n(B) \cong 0$ for all $n \in N$. This means that

$$0 = \mathrm{im}\Psi = \ker \partial_k$$
$$H_k(\mathbb{S}^{n-1}) = \ker \Phi = \mathrm{im}\partial_{n+1}.$$

We get that the maps $\partial \colon H_k(\mathbb{S}^n) \to H_{k-1}(\mathbb{S}^{n-1})$ are isomorphisms. And thus with the fact that $H_1(\mathbb{S}^1) \cong \mathbb{Z}$ the claim is proven by induction on k.[5] □

[5] The fact that the first homology group of $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$ can easily be proven with another type of homology, namely simplicial homology [4, p. 106]. And since the singular and simplicial homology groups are isomorphic for triangulable spaces it can be deduced that $H_1(\mathbb{S}^1) \cong \mathbb{Z}$ also in the singular case. A triangulation of $\mathbb{S}^1$ is the circle graph with three vertices.

4 L_0-Groups, connection between extreme amenability and chromatic numbers

In this chapter, we want to establish a connection between the extreme amenability of a L_0-group and the boundedness of the chromatic numbers of a sequence of graphs. These graphs will be constructed using the structure of the L_0-groups.

Firstly we have to define what is meant by the symbol $L_0(\mu, G)$. Some authors already consider the set of simple functions from Defintion 2.40 together with the operation $\star$ defined in Theorem 2.41 as the topological group $L_0(\mu, G)$. In other works such as [15] this topological group is the Raikov completion (see [1, Chapter 3.6]) of $S(\mu, G)$. In this case the topological group $S(\mu, G)$ would be a dense subgroup of $L_0(\mu, G)$.

When talking about extreme amenability of topological groups it is sufficient to prove it for dense subgroups because of the following Theorem.

Theorem 4.1:
Let G be a topological group and let $H \subseteq G$ be a dense subgroup. Then H is extremely amenable if and only if G is extremely amenable.

Proof. Let $g \in G$ and let X be a compact space. Take a continuous action

$$G \times X \to X,\ (g, x) \mapsto g \centerdot x.$$

This action restricts to a continous action on H. Since H is dense in G, by Theorem 2.20 we can find $\mathcal{G} \in \mathrm{Flt}(H)$ that converges to g. Now let $\mathcal{F} \in \mathrm{UFlt}(X)$ with $\mathcal{G} \subseteq \mathcal{F}$. This exists due to Theorem 2.23 and this filter still converges to g. Furthermore since H is extremely amenable there exists $x_0 \in X$ such that $h \centerdot x_0 = x_0$. Now consider the map f defined as

$$f \colon G \to X,\ g \mapsto g \centerdot x_0.$$

It follows that

$$\forall F \in \mathcal{F} \colon f(F) = \{x_0\}$$

and thus

$$g \centerdot x_0 = f(g) = \lim_{F \to \mathcal{F}} f(F) = x_0.$$

The exchange of function application and limit is possible since f is continuous by definition. Then for the other implication assume that the subgroup is not extremely amenable H. This means there exists an continuous action $H \times X \to X$ which admits no fixed point. It is possible to extend the action in a continuous way to the whole group G like above. Then we have constructed a continuous action on G with no fixed point. □

This result allows us in the main results of the thesis to only talk about the simple function because the extreme amenability property is inherited by $L_0(\mu, G)$ from them.

In the following the symbol $\bar{1}$ represents the constant function which sends each element of X to 1 in the set $S(\mu, \mathbb{Z})$.

Definition 4.2:

Let X be a set and μ be a submeasure on X defined on a subalgebra $\mathcal{B} \subseteq \mathfrak{P}(X)$. Addtionally let $\varepsilon \in \mathbb{R}_{>0}$ and $\mathcal{P} \in \Pi(\mathcal{B})$ then define the graph $\Gamma(\varepsilon, \mathcal{P}, \mu)$ in the following way

- the vertex set of $\Gamma(\varepsilon, \mathcal{P}, \mu)$ is $\mathbb{Z}^{\mathcal{P}}$,
- the vertices $f, g \in \mathbb{Z}^{\mathcal{P}}$ are connected in $\Gamma(\varepsilon, \mathcal{P}, \mu)$ if

$$\mu(\{P \in \mathcal{P} \colon f(P) \neq g(P) + 1\}) < \varepsilon.$$ ♠

Lemma 4.3:

Let X be a set and let μ be a submeasure on a subalgebra $\mathcal{B} \subseteq \mathfrak{P}(X)$. Additionally, let $\mathcal{P} \in \Pi(\mathcal{B})$ and $\varepsilon \in \mathbb{R}_{>0}$ then for $f, g \in \mathbb{Z}^{\mathcal{P}}$ with $f - g \in \bar{1} + V_\varepsilon$ it follows that f and g are connected in $\Gamma(\varepsilon, \mathcal{P}, \mu)$ by an edge.

Proof.

$$\begin{aligned} f - g \in \bar{1} + V_\varepsilon &\iff \mu(\{x \in X \colon (f-g)(x) \neq 1\}) < \varepsilon \\ &\iff \mu(\{x \in X \colon f(x) \neq g(x) + 1\}) < \varepsilon \\ &\iff \mu\left(\bigcup \{A \in \mathcal{P} \colon f(A) \neq g(A) + 1\}\right) < \varepsilon. \end{aligned}$$ □

Theorem 4.4 (Sabok, [13, Lemma 5])**:**
Let X be a set, $\mu\colon \mathcal{B} \to [0,\infty)$ a diffuse submeasure on X where $\mathcal{B}$ is a subalgebra of $\mathfrak{P}(X)$. Then the following are equivalent:

1. $S(\mu, G)$ is extremely amenable for every non-trivial, Hausdorff, abelian topological group G,

2. $\forall \varepsilon > 0\colon \sup\limits_{\mathcal{P}\in\Pi(\mathcal{B})} \chi(\Gamma(\varepsilon, \mathcal{P}, \mu)) = \infty.$

Proof. (1) $\Rightarrow$ (2): Let $\varepsilon \in \mathbb{R}_{>0}$ and assume that there exists $d \in \mathbb{N}$ such that

$$\forall \mathcal{P} \in \Pi(\mathcal{B})\colon \chi(\Gamma(\varepsilon, \mathcal{P}, \mu)) \leq d.$$

Given this assumption we claim that the the group $S(\mu, \mathbb{Z})$ is not extremely amenable. To this end choose a coloring of $\Gamma(\varepsilon, \mathcal{P}, \mu)$ called $c_{\mathcal{P}}\colon \mathbb{Z}^{\mathcal{P}} \to \{1, \ldots, d\}$ for every $\mathcal{P} \in \Pi(\mathcal{B})$. Now the goal is to define a coloring on the whole set $S(\mu, \mathbb{Z})$. To this end define the family of sets

$$\mathcal{A} := \{\{\mathcal{P} \in \Pi(X)\colon \mathcal{P}_0 \preccurlyeq \mathcal{P}\}\colon \mathcal{P}_0 \in \Pi(X)\}.$$

This is a well-defined filter basis by Theorem A.5 and Lemma 2.15. From Theorem 2.23 it follows that there is an ultrafilter $\mathcal{U}$ containing the filter $\mathcal{F}(\mathcal{A})$. Let $f \in S(\mu, \mathbb{Z})$. Define $f_{\mathcal{P}} \in \mathbb{Z}^{\mathcal{P}}$ for all finite partitions $\mathcal{P}$ refining the partion $\mathcal{P}_f$ as

$$f_{\mathcal{P}}(A) = k \iff A \subseteq f^{-1}(\{k\}).$$

This functions is well-defined because of the definition of refinment of partitions and thus also $f_{\mathcal{P}} \in S(\mu, \mathbb{Z})$. Now define a coloring $c\colon S(\mu, \mathbb{Z}) \to \{1, \ldots, d\}$ of $S(\mu, \mathbb{Z})$ as the limit over the ultrafilter

$$c(f) := \lim_{\mathcal{P}\to\mathcal{U}} c_{\mathcal{P}}(f_{\mathcal{P}}).$$

Convergence of the limit above is ensured by the fact that it is a limit in a finite and discrete space and thus the limit has to take one and only one value in $\{1, \ldots, d\}$.
Let $X_i := \{f \in S(\mu, \mathbb{Z})\colon c(f) = i\}$ for each $i = 1, \ldots, d$ which is a cover of $S(\mu, \mathbb{Z})$ and let $f, g \in \mathbb{Z}^{\mathcal{P}}$. If $f - g \in V_\varepsilon(\bar{1})$ then by Lemma 4.3 it follows that $c_{\mathcal{P}}(f) \neq c_{\mathcal{P}}(g)$.

In the limit of the ultrafilter $\mathcal{U}$ we get $V_\varepsilon(\bar{1}) \cap (X_i - X_i)$ and thus $X_i - X_i$ is not dense at $\bar{1}$ for each $i = 1, \dots, d$. By Pestovs characterization [10, Theorem 3.4.9] of extreme amenability it follows that $S(\mu, \mathbb{Z})$ is not extremely amenable.

(2) $\Rightarrow$ (1): For the second implication suppose that $S(\mu, G)$ is not extremely amenable. By Pestovs characterization [10, Theorem 3.4.9] we get the existence of a set $S \subseteq S(\mu, G)$ that is big on the left[6] such that $S(\mu, G) \neq \overline{S - S}$. Let $E \subseteq S(\mu, G)$ finite with an enumeration $E = \{e_1, e_2, \dots, e_n\}$ with $n \in \mathbb{N}$ such that $S(\mu, G) = E + S$ and let $f \in S(\mu, G) \setminus \overline{S - S}$. It follows that there exist $W \in \mathcal{N}_G(e_G)$ and $\varepsilon \in \mathbb{R}_{>0}$ such that

$$W_\varepsilon(f) \cap (S - S) = \emptyset$$

since f is not in the closure of $S - S$. Now let $V_\varepsilon := \{h \in S(\mu, G) \colon \mu(\{x \in X \colon h(x) \neq e_G\}) < \varepsilon\}$. Since $e_G \in W$ it follows that $V_\varepsilon + f \subseteq W_\varepsilon(f)$ and hence

$$(V_\varepsilon + f) \cap (S - S) = \emptyset.$$

Define the partition $\mathcal{P}_f$ induced by f. Now let $\mathcal{P} \in \Pi(\mathcal{B})$ with $\mathcal{P}_f \preccurlyeq \mathcal{P}$ and let $k \in \mathbb{Z}^{\mathcal{P}}$. Define $S_i := e_i + S$ for $i = 1, \dots, n$ and

$$g_k \colon X \to G,\; x \mapsto k(\iota_{\mathcal{P}}(x)) f(x)$$

which is an element of $S(\mu, G)$. Now we can define the mapping c as

$$c \colon \mathbb{Z}^{\mathcal{P}} \to \{1, \dots, d\},\; k \mapsto i \iff g_k \in S_i.$$

Claim: The mapping c is a coloring of the graph $\Gamma(\varepsilon, \mathcal{P}, \mu)$. To this end suppose there are two nodes $k, l \in \mathbb{Z}^{\mathcal{P}}$ which are connected and have the same color $i \in \{1, \dots, d\}$. Let $B := \bigcup\{A \in \mathcal{P} \colon k(A) \neq l(A) + 1\}$. It holds that $\mu(B) \leq \varepsilon$ since k and l are connected in $\Gamma(\varepsilon, \mathcal{P}, \mu)$. For $x \in X \setminus B$ it holds that

$$g_k(x) - g_l(x) = k(\iota_{\mathcal{P}}(x)) f(x) - l(\iota_{\mathcal{P}}(x)) f(x) = (k(\iota_{\mathcal{P}}(x)) - l(\iota_{\mathcal{P}}(x)) f(x) = f(x)$$

[6] This means that there exists a finite set $E \subseteq S(\mu, G)$ such that $S(\mu, G) = E + S$. Additionally $\{e + S \colon e \in E\}$ is a covering of $S(\mu, G)$.

since $k(A) = l(A) + 1$ for $x \notin B$. It follows that

$$\mu(\{x \in X \colon g_k(x) - g_l(x) \neq f(x)\}) < \varepsilon.$$

which means $g_k - g_l \in f + V_\varepsilon$. Furthermore we have $g_k, g_l \in S_i$ because of the definition of c. Hence $g_k - g_k \in (f + V_\varepsilon) \cap (S_i - S_i)$ ↯. This proves the claim that c is a coloring of $\Gamma(\varepsilon, \mathcal{P}, \mu)$ with d colors. $\square$

5 The Borsuk-Ulam Theorem and a generalization

The following section describes the Borsuk-Ulam Theorem and an important generalization by Volovikov needed to prove the following claim.

Theorem 5.1:
If $p \in \mathbb{P}$ and $n, l \in \mathbb{N}$ with $n \geq l$ such that

$$d(p-1) \leq l-1,$$

then for every continuous map $f \colon \|K_p(n,l)\| \to \mathbb{R}^d$ there is a point in $\|K_p(n,l)\|$ whose $\mathbb{Z}_p$-Orbit is mapped to a single point in $\mathbb{R}^d$ by f.

This Theorem will be needed later to prove a bound on the chromatic numbers of the graphs $\Gamma(\mu, \mathcal{P}, \varepsilon)$. An important tool needed in the proof of Theorem 5.1 is a result of Volovikov which he described in [17]. It is a generalization of the Borsuk-Ulam Theorem.

Lemma 5.2:
Let X be a connected paracompact Hausdorff space, acted on without fixed points by a cyclic group $\mathbb{Z}_p$ of prime order p. For any continuous function $f \colon X \to M$ and generator T of $\mathbb{Z}_p$ let

$$A(f) := \{x \in X \colon f(x) = f(Tx) = \cdots = f(T^{p-1}x)\}$$

be the set of points of which the $\mathbb{Z}_p$-orbit is mapped by f to a single point. Suppose that $\tilde{H}^i(X; \mathbb{Z}_p) = 0$ for $0 < i < n$ and M is a compact $\mathbb{Z}_p$ orientiable topological manifold of dimension m. If the map $\tilde{H}^n(f) \colon \tilde{H}^n(X; \mathbb{Z}_p) \to \tilde{H}^n(M; \mathbb{Z}_p)$ has zero image, then the cohomological dimension over $\mathbb{Z}_p$ of $A(f)$ is at least $n - m(p-1)$.

In his book "Using the Borsuk-Ulam Theorem" [6] Matoušek compiled various formulations of the Borsuk-Ulam Theorem and described it can be generalized with the concept of $\mathbb{Z}_2$ spaces and more generally with E_nG spaces where G is a group. In our case $G = \mathbb{Z}_p$ and instead of two there will be $p \in \mathbb{P}$ points that get mapped to the same point by any continuous function.

5.1 The simplicial complex $K_p(n,l)$

Definition 5.3:
Let $n, p \in \mathbb{N}$, then $\tau \subseteq [n+1] \times [p]$ is called a **partial function** if

$$\tau\Big|_{\mathrm{dom}(\tau)} = \tau \cap (\mathrm{dom}(\tau) \times [p])$$

is a function. Let τ_1, τ_2 be partial functions from $[n+1]$ to $[p]$. Now define

$$\tau_1 \subseteq_f \tau_2 :\Longleftrightarrow \mathrm{dom}(\tau_1) \subseteq \mathrm{dom}(\tau_2) \ \wedge\ \forall k \in \mathrm{dom}(\tau_1) \colon \tau_1(k) = \tau_2(k).$$

If a partial function τ is defined on every element of the domain $[n+1]$ then it is called a **total function**. The set of all partial functions from a set X to a set Y will be denoted by $P(X,Y)$.

A **component interval** of τ is any maximal interval $I \subseteq [n+1]$ such that τ is constant on $I \cap \mathrm{dom}(\tau)$. Define $\mathbb{I}(\tau)$ to be the set of component intervals of τ.

In addition to that let f, g be partial functions from the set X to set Y with $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \emptyset$ and define

$$f \uplus g : \mathrm{dom}(f) \cup \mathrm{dom}(g) \to Y,\ x \mapsto \begin{cases} f(x),\ x \in \mathrm{dom}(f), \\ g(x),\ x \in \mathrm{dom}(g). \end{cases} \quad \spadesuit$$

Definition 5.4:
Let $l \in \mathbb{N}$ and $p \in \mathbb{P}$. Let $V_p(n,l) = P([n+1],[p])$ with the additional properties

i.) $\forall \tau \in V_p(n,l) \colon n - l \leq |\mathrm{dom}(\tau)|$,

ii.) $\forall \tau \in V_p(n,l) \colon\ |\mathbb{I}(\tau)| \leq l+1$. $\spadesuit$

Definition 5.5:
Let $K_p(n,l)$ be the abstract simplicial complex with

i.) $V(K_p(n,l)) = V_p(n,l)$,

ii.) the simplices in $K_p(n,l)$ are chains of $V_p(n,l)$ with respect to $\subseteq_f$.

This complex is a $\mathbb{Z}_p$ complex with the action

$$\mathbb{Z}_p \times V_p(n,l) \to V_p(n,l),\ (k,\tau) \mapsto \lambda(k,\tau) = k +_f \tau,$$

with $\mathrm{dom}(k +_f \tau) = \mathrm{dom}(\tau)$ and $\forall i \in \mathrm{dom}(k +_f \tau)\colon (k +_f \tau)(i) = k +_p \tau(i)$. ♠

Definition 5.6:

Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$. Define $S_p(n)$ as the abstract simplicial complex with

i.) $V(S_p(n)) = \{\tau\colon [n+1] \to [p]\colon |\mathrm{dom}(\tau)| = 1\}$,

ii.) $\sigma \in S_p(n) \setminus \emptyset \;:\Longleftrightarrow\; \forall \tau_1, \tau_2 \in \sigma\colon\ (\tau_1 \neq \tau_2) \Rightarrow (\mathrm{dom}(\tau_1) \cap \mathrm{dom}(\tau_2) = \emptyset)$. ♠

Definition 5.7:

Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$. For $a \in (\mathbb{Z}_p \setminus \{0\})^{n+1}$ define $S_p(n,a)$ as

$$S_p(n,a) := \{0, a(0)\} \star \{0, a(1)\} \star \ldots \star \{0, a(n)\}.$$ ♠

Remark 5.8:

By Lemma 3.19 it holds that $\|S_p(n,a)\| \cong \mathbb{S}^n$ for $n \in \mathbb{N}$.

Now, our goal is to show that the complex has trivial reduced cohomology groups $\tilde{H}^i(\|K_p(n,l)\|; \mathbb{Z}_p)$ with coefficients in $\mathbb{Z}_p$ such that we can apply Lemma 5.2. Since $\mathbb{Z}_p$ together with $+_p$ and the multiplication of the natural numbers $\bmod p$ is a field, it is sufficient to show that the reduced homology groups $\tilde{H}_i(K_p(n,l); \mathbb{Z}_p)$ are trivial. The triviality of the cohomology groups follows from the Universal Coefficient Theorem **??**.

Lemma 5.9:

Let $n \in \mathbb{N}$ and let $p \in \mathbb{P}$. Then

$$\|K_p(n,n)\| \;\simeq\; \bigvee_{i=1}^{(p-1)^{n+1}} \mathbb{S}^n.$$

Proof. First notice that $\mathbb{Z}_p \cong \bigvee_{i=1}^{p-1} \mathbb{S}^0 =: S$ with basepoint 1 via the isomorphism

$$V(\mathbb{Z}_p) \to V(S),\ k \mapsto \begin{cases} [(1,1)]_\sim,\ k = 0, \\ [(-1,k)]_\sim,\ k \neq 0. \end{cases}$$

Furthermore it holds that $\|S_p(n)\| \simeq \|\bigvee_{i=0}^{n} \mathbb{Z}_p\| \simeq \bigvee_{i=0}^{n} \bigvee_{j=1}^{p-1} \mathbb{S}^0$ and thus

$$\|S_p(n)\| \simeq \left\| \bigvee_{a \in (\mathbb{Z}_p \setminus 0)^{n+1}} S_p(n,a) \right\| \overset{5.8}{\simeq} \bigvee_{a \in (\mathbb{Z}_p \setminus 0)^{n+1}} \mathbb{S}^n.$$

It is clear that $V(S_p(n)) \cong V_p(n,n)$ because the vertices in $\mathrm{sd}(S_p(n))$ are maximal chains in $S_p(n)$ which are the total functions from $[n+1]$ to $[p]$ by definition and thus $K_p(n,n) \cong \mathrm{sd}(S_p(n))$. By Lemma 3.13 it follows that

$$\|K_p(n,n)\| \simeq \bigvee_{a\in(\mathbb{Z}_p\setminus 0)^{n+1}} \mathbb{S}^n \simeq \bigvee_{i=1}^{(p-1)^{n+1}} \mathbb{S}^n$$

since $|(\mathbb{Z}_p \setminus \{0\})^{n+1}| = (p-1)^{n+1}$. □

In the following it will be needed to show that $K_p(n,n)$ also has a wedge-decomposition similiar to $S_p(n)$.

Definition 5.10:

Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$. Define the abstract simplicial complex $K_p(n,a)$ for $a \in (\mathbb{Z}_p \setminus \{0\})^{n+1}$ as

$$K_p(n,a) = \{\tau \in K_p(n,n) \colon \forall k \in \mathrm{dom}(\tau) \colon \tau(k) = 0 \ \vee \ \tau(k) = a(k)\}. \qquad \spadesuit$$

Remark 5.11:

Let $n, l \in \mathbb{N}$ with $n \geq l$, $p \in \mathbb{P}$ and $a \in (\mathbb{Z}_p \setminus \{0\})^{n+1}$. It holds that

$$\|K_p(n,a)\| \cong \|\mathrm{sd}(S_p(n,a))\|,$$
$$\|K_p(l,l)\| \cong \|\mathrm{sd}(S_p(l))\|.$$

Proof. Rememeber that the vertices of $\mathrm{sd}(S_p(n,a))$ are the total chains in $S_p(n,a)$ which are isomorphic to $V_p(n,n)$ which means they represent total functions which are either 0 or have the same value as the function a which is the definition of an element in $K_p(n,a)$. The same argument can be applied for the second claim $\|K_p(l,l)\| \cong \|\mathrm{sd}(S_p(l))\|$. □

Now the following decomposition of $K_p(n,n)$ can be proven:

Lemma 5.12:

Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$. Then it follows that

$$\|K_p(n,n)\| \simeq \left\| \bigvee_{a\in(\mathbb{Z}_p\setminus\{0\})^{n+1}} K_p(n,a) \right\|.$$

Proof. Define D_n as a simplicial complex with

- $V(D_n) = P([m], 0)$,
- the simplices are chains with respect to $\subseteq_f$,

and define $\bar{0}$ as the total, constant zero function. Note that $D_n \subseteq K_p(n,n)$ and that $D_n \subseteq K_p(n,a)$ for all $a \in (\mathbb{Z}_p \setminus \{0\})^{n+1}$ by definition. Also note that $\bigvee_{a\in(\mathbb{Z}_p\setminus\{0\})^{n+1}} K_p(n,a) =: K$ contains $(p-1)^{n+1}$ copies of D_n glued together at $\bar{0}$ and $K_p(n,n)$ contains one copy of D_n. D_n is contractible since it is the barycentric subdivision of $\{0\}^{*(n+1)}$ which is the $(n+1)$-fold join of the one point space which is the same as taking $n+1$ nested cones of this space. Since taking a cone always produces a contractible space this space is contractible and the barycentric subdivision keeps the homotopy type. Now collapse all copies of D_n in K and collapse the one copy in $K_p(n,n)$ to $\bar{0}$. Then the two sets $K_p(n,n)$ and K are isomorphic by sending a $\tau \in V(K_p(n,n))$ to the component of K where $\tau \subseteq_f a$. Since the simplices in $K_p(n,n)$ are chains with respect to $\subseteq_f$ this means that if the greatest element of a chain lies in the component $K_p(n,a)$ for some a in K the whole chain is contained in this component meaning the simplices are maintained by this isomorphism. □

Definition 5.13:
Define $L_p(n,l)$ as a subcomplex of $K_p(n,n)$ with

$$\forall \tau \in V(L_p(n,l))\colon |\mathbb{I}(\tau)| \leq l+1$$

and define $J_p(n,l)$ as a subcomplex of $K_p(n,l)$ with

$$\forall \tau \in V(J_p(n,l))\colon |\mathrm{dom}(\tau)| \geq n-l. \qquad \spadesuit$$

Now note that $K_p(n,l) = J_p(n,l) \cap L_p(n,l)$. We can use this fact for applying the Mayer-Vietoris-Sequence for finding the homology groups of $\|K_p(n,l)\|$. This means we first have to study the complexes $J_p(n,l)$ and $L_p(n,l)$ and show that their homology groups are trivial up to some point.

We start our analysis with the subcomplex $L_p(n,l)$. Notice that if we compare $K_p(l,l)$ to $L_p(n,l)$ than $L_p(n,l)$ contains additional partial functions namely all the functions with a domain not contained in $[l+1]$. We will show that we can extend the complex $K_p(l,l)$ by adding cones to specific subcomplexs such that we get a new complex that is isomorphic to $L_p(n,l)$. The intuitition behind this idea is that the added cone points and the join of already existing partial functions will act like the missing functions with large domain.

Definition 5.14:

Let $\mathcal{K}$ be a simplicial complex, $p, m \in \mathbb{N}$ and let $(L_\sigma)_{\sigma \in P([m],[p])}$ be a family of subcomplexes of $\mathcal{K}$ such that $L_\sigma \subseteq L_\tau$ if $\tau \subseteq \sigma$.

Define the complex $\mathrm{Cone}(\mathcal{K}, (L_\sigma)_{\sigma \in P([m],[p])})$ inductively. Firstly let $\mathcal{K}^0 = \mathcal{K}$ and let $L^0_\sigma = L_\sigma$ for all $\sigma \in P([m],[p])$. Now define X^k and L^k_σ by induction on $k \leq m$ for $\sigma \in P([k+1,m],[p])$ as

$$\begin{aligned} \mathcal{K}^{k+1} &= \mathrm{Cone}(\mathcal{K}^k, (L^k_{(k,i)})_{i \in [p]}), \\ L^{k+1}_\sigma &= \mathrm{Cone}(L^k_\sigma, (L^k_{(k,i) \uplus \sigma})_{i \in [p]}) \end{aligned}$$

where (j,i) denotes the partial function $\tau \colon [m] \to [p]$ with $\mathrm{dom}(\tau) = \{j\}$ and $\tau(j) = i$. Define $\mathrm{Cone}(K, (L_\sigma)_{\sigma \in P([m],[p])}) =: \mathcal{K}^m$. Additionally this definition also works for a set $Y \subseteq [m]$ and all $\sigma \in P(Y,[p])$ only looking at partial functions with a resticted domain because Y inherits the natural order of $\mathbb{N}$. ♠

The well-definedness of the induction step follows from the assumption that $A_\sigma \subseteq A_\tau$ if $\tau \subseteq \sigma$.

Lemma 5.15:

Let $\mathcal{K}$ be a simplicial complex and let $(L_\sigma)_{\sigma \in P(m,p)}$ be a sequence of subcomplexes for $p, m \in \mathbb{N}$ empty or contractible then

$$\|\mathcal{K}\| \simeq \|\mathrm{Cone}(\mathcal{K}, (L_\sigma)_{\sigma \in P(m,p)})\|.$$

Proof. We prove the claim by induction. So let $k = 0$. Then the claim follows directly from Lemma 3.18 since the sequence of subcomplexes is finite. Suppose that the claim holds for an arbitrary $k \leq m-1$. Then since the L^k_σ are all empty or contractible by Lemma 3.18 we can deduce that X^{k+1} is homotopy equivalent to X^0. □

Lemma 5.16:

For $n, l \in \mathbb{N}$ with $n \geq l$ and $p \in \mathbb{P}$ it holds that

$$\|L_p(n,l)\| \simeq \|\mathrm{Cone}(K_p(l,l), (L_\sigma)_{\sigma \in P([l+1,n+1],[p])})\|$$

where L_σ is defined as

$$L_\sigma = \{\tau \in V_p(l,l)\colon\ |\mathbb{I}(\tau \uplus \sigma)| \leq l+1\}.$$

Proof. For $n = l$ this is trivial. So assume that $n > l$ and let

$$K := \mathrm{Cone}(K_p(l,l), (L_\sigma)_{\sigma \in P([l+1,n+1],[p])}).$$

We will construct a combinatorial isomorphism $\Phi\colon V(L_p(n,l)) \to V(K)$ which proves that claim of the Lemma since the geometric realizations of the two complexes will be homeomorphic.

Firstly notice that all $\tau \in V_p(l,l) \subseteq L_p(n,l)$ are already contained in $K_p(l,l)$. This means that

$$\Phi|_{V_p(l,l)} = \mathrm{id}_{V_p(l,l)}.$$

Denote by $\Delta_{(k,i)}$ the cone point over the complex $L^k_{(k,i)}$. Define $\Phi((k,i)) = \Delta_{(k,i)}$ for $k \in [l+1, n+1]$ which means that this cone point will act as the function (k,i) in the complex K.

We prove that Φ is well-defined by induction over m from $l+1$ to n. Let $m = l+1$ and consider $\tau \in L_p(n,l)$ where $\tau \in P([l+2],[p])$. If $l+1 \notin \mathrm{dom}(\tau)$ then $\Phi(\tau) = \tau \in K_p(n,l)$. Otherwise consider the two cases

- $\Phi(\tau) = \tau' \sqcup \Delta_{(l+1,i_1)}$ for $\tau' \in V_p(l,l)$, and because of the definition of $L_p(n,l)$ we know that $|\mathbb{I}(\tau)| = |\mathbb{I}(\tau' \uplus (l+1, \tau(l+1)))| \leq l+1$ which means $\tau' \in L^{l+1}_{(l+1,\tau(l+1))}$ and thus $\Phi(\tau) \in \mathrm{Cone}(L^{l+1}_{(l+1,\tau(l+1))})$,
- $\Phi(\tau) = \Delta_{(l+1,\tau(l+1))}$ which is trivially in $\mathrm{Cone}(L^{l+1}_{(l+1,\tau(l+1))})$.

Now assume that Φ is well-defined for an $m < n$ and consider the case $m+1$. Let $\tau \in L_p(n,l)$ with $\tau \in P([m+2, n+1])$. If $m+1 \notin \mathrm{dom}(\tau)$ then the Φ is well-defined by applying the induction hypothesis. Otherwise consider the cases

- $\Phi(\tau) = \tau' \sqcup \Delta_{(m+1,i_m)}$ for $\tau' \in V_p(m,l)$. On the one hand, we know that $\Phi(\tau') \in L^m_{(m,\tau'(m))}$ which means that $\tau' \in V_p(l,l)$. On the other hand, we know that $|\mathbb{I}(\tau)| = |\mathbb{I}(\tau' \uplus (m+1, \tau(m+1)))| \le l+1$ and thus $\tau' \in L^m_{(m,\tau'(m))\uplus(m+1,\tau(m+1))}$ which implies $\tau' \in L^{m+1}_{(m+1,\tau(m+1))}$. It follows that $\Phi(\tau) \in \mathrm{Cone}(L^{m+1}_{(m+1,\tau(m+1))})$,
- $\Phi(\tau) = \Delta_{(m+1,\tau(m+1))}$ which is trivially in $\mathrm{Cone}(L^{m+1}_{(m+1,\tau(m+1))})$.

Thus Φ is well-defined for all $\tau \in L_p(n,l)$.

Let $\tau, \tau' \in L_p(n,l)$ and assume that $\Phi(\tau) = \Phi(\tau')$. If $\tau, \tau' \in K_p(l,l)$ then

$$\tau = \Phi(\tau) = \Phi(\tau') = \tau'.$$

Now assume that τ, τ' are both not in $K_p(l,l)$ and $\tau, \tau' \in V_p(n,l)$. First assume that $\mathrm{dom}(\tau) \neq \mathrm{dom}(\tau')$. Then w.l.o.g. we can assume that there is a $k \in [n+1]$ such that $k \in \mathrm{dom}(\tau)$ and $k \notin \mathrm{dom}(\tau')$. This means that $\Delta_{(k,\tau(k))} \in \Phi(\tau)$ and $\Delta_{(k,\tau(k))} \notin \Phi(\tau')$ which contradicts the assumption $\Phi(\tau) = \Phi(\tau')$ so their domains must be equal.

Now write $\mathrm{dom}(\tau) = I \cup J$ with $I \subseteq [l+1]$ and $J \subseteq [l+1, n+1]$. Define $J = \{k_1, \ldots, k_r\}$ for $r \le n-l$ and consider

$$\Phi(\tau) = \tau_1 \sqcup (k_1, \tau(k_1)) \sqcup \cdots \sqcup (k_r, \tau(k_r)),$$
$$\Phi(\tau') = \tau_2 \sqcup (k_1, \tau'(k_1)) \sqcup \cdots \sqcup (k_r, \tau(k_r)),$$

where $\tau_1 = \tau|_I$ and $\tau_2 = \tau'_I$. Since the images of the functions are equal we get that $\tau_1 = \tau_2$ which means $\tau|_I = \tau'|_I$ and that $\tau(k_i) = \tau'(k_i)$ for all $i \in J$. This means that $\tau = \tau'$ and thus the function Φ is injective.

Now take $\sigma \in K$. If $\sigma \in K_p(l,l)$ then $\sigma = \tau \in K_p(l,l)$ and since $K_p(l,l) \subseteq L_p(n,l)$ we get that $\Phi(\tau) = \tau = \sigma$. Now assume that $\tau \notin K_p(l,l)$. By the inductive definition of K we know that we can write σ as

$$\sigma = \tau' \sqcup \Delta_{(k_1,i_1)} \sqcup \cdots \sqcup \Delta_{(k_r,i_r)}$$

for $k_j \in [l+1, n+1]$ and $i_j \in [p]$ for all $j = 1, \ldots, r \le n-l$ such that $k_{j_1} < k_{j_2}$

for $j_1 < j_2$. Then

$$\tau' \sqcup \Delta_{(k_1,i_1)} \sqcup \cdots \sqcup \Delta_{(k_{r-1},i_{r-1})} \in L^{k_r}_{k_r,i_r}$$

and by repeated application of the definition we get that

$$\tau' \in L_{(k_1,i_1)\uplus\cdots\uplus(k_r,i_r)}.$$

This means

$$\tau := \tau' \uplus (k_1, i_1) \uplus \cdots \uplus (k_r, i_r) \in L_p(n, l)$$

and $\Phi(\tau) = \sigma$ and thus Φ is surjective.

It remains to show that Φ is a simplicial map. So let $\sigma \in L_p(n, l)$. By definition this is a chain of partial functions with repect to $\subseteq_f$. If this chain is contained in $K_p(l, l)$ then $\Phi(\sigma) = \sigma$ and thus a simplex in $K_p(l, l)$. If not then we know that

$$\tau' \sqcup \Delta_{(k_1,i_1)} \sqcup \cdots \sqcup \Delta_{(k_{r-1},i_{r-1})} \subseteq \tau' \sqcup \Delta_{(k_1,i_1)} \sqcup \cdots \sqcup \Delta_{(k_{r-1},i_{r-1})} \sqcup \Delta_{(k_r,i_r)} \tag{3}$$

for $\tau' \in V_p(l, l) \cup \{\emptyset\}$ and $k_j \in [l + 1, n + 1]$, $i_j \in [p]$ for all $j = 1, \ldots, r \leq n - l$. This fact and the definition of K imply that the image of an arbitrary simplex $\sigma \in L_p(n, l)$ is a simplex in K. Now consider that inverse map Φ^{-1}. We can see that this is a simplicial map by Equation 3 and the fact that by definition of K there are no simplices in K which simultaneously contain σ_1 and σ_2 of the form

$$\begin{aligned}
\sigma_1 &= \tau' \sqcup \Delta_{(k_1,i_1)} \sqcup \cdots \sqcup \Delta_{(k_s,i_s)} \sqcup \cdots \sqcup \Delta_{(k_{r-1},i_{r-1})}, \\
\sigma_2 &= \tau' \sqcup \Delta_{(k_1,i_1)} \sqcup \cdots \sqcup \Delta_{(k_s,i'_s)} \sqcup \cdots \sqcup \Delta_{(k_{r-1},i_{r-1})} \sqcup \Delta_{(k_r,i_r)},
\end{aligned}$$

where τ', k_j and i_j are as above with $i_s \neq i'_s$. $\square$

Lemma 5.17:
For each $\sigma \in P([l + 1, n + 1], [p])$ the set L_σ is either empty or contractible.

Proof. Prove by induction on $\mathbb{N} \ni i \leq l + 1$ that for $\rho \in [p]^{[i+1,l+1]}$ with $l - i$ component intervals and $\rho(l) \neq \sigma(l + 1)$ that the set

$$L_{\sigma,\rho} = \{\tau \in V_p(i, l) \colon\ |\mathbb{I}(\tau \uplus \rho \uplus \sigma)| \leq l + 1\}$$

is either empty or contractible. Note that the set is well-defined since in the definition $\mathrm{dom}(\tau) \subseteq [i+1]$, $\mathrm{dom}(\rho) = [i+1, l+1]$ and $\mathrm{dom}(\sigma) \subseteq [l+1, n+1]$. Additionally if $i = l$ then $L_{\sigma,\rho} = L_{\sigma}$.

Start with $i = 0$. Notice that $|\mathbb{I}(\rho \uplus \sigma)| \geq l$ since $\rho(l) \neq \sigma(l+1)$. Then there are two cases

- $|\mathbb{I}(\rho \uplus \sigma)| > l+1$ which means that $L_{\rho,\sigma} = \emptyset$,
- $|\mathbb{I}(\rho \uplus \sigma)| = l+1$ which means that $L_{\rho,\sigma}$ contains the function $\tau(0) = \rho(1)$ and the empty function and is thus contractible as a one point space.

We consider the induction step $i-1$ to i. Define for $\mathbb{N} \ni j < p$ the total function $\tau_j \colon \{i\} \to \{j\}$, $x \mapsto j$ and define

$$B = \{\tau \in V_p(i-1, l) \colon\ |\mathbb{I}(\tau \uplus \rho \uplus \sigma)| \leq l+1\}$$

and

$$B_j = \{\tau \in V_p(i-1, l) \colon\ |\mathbb{I}(\tau \uplus \tau_j \uplus \rho \uplus \sigma)| \leq l+1\}.$$

Firstly B_j is well-defined since $\mathrm{dom}(\tau) \subseteq [i]$ and $\mathrm{dom}(\rho) \subseteq [i+1]$ and thus

$$\mathrm{dom}(\tau) \cap \mathrm{dom}(\tau_j) \cap \mathrm{dom}(\rho) = \emptyset.$$

Next notice that $B_{j_0} = B$ because $\tau_{j_0}(i) = \rho(i+1)$ and that B_j are either contractible or empty by the induction hypothesis.

There are again two cases

- $|\mathbb{I}(\rho \uplus \sigma)| > l+1$ which means that $L_{\sigma,\rho}$ is empty,
- $|\mathbb{I}(\rho \uplus \sigma)| \leq l+1$ then $L_{\rho,\sigma} \cong \mathrm{Cone}(B, (B_j)_{j\in[p]})$ as sets and by Lemma 5.15 we get that the realization of $\mathrm{Cone}(B, (B_j)_{j\in[p]})$ is homotopy equivalent to the geometric realization of $B \cup \mathrm{Cone}(B_{j_0})$ and since $B = B_{j_0}$ this is the same as $\mathrm{Cone}(B)$ which is contractible.

The map between $\mathrm{Cone}(B, (B_j)_{j\in[p]}) \to L_{\rho,\sigma}$ mentioned above sends each $\tau \in B$ to $\tau \in V_p(i-1, l) \cap L_{\rho,\sigma}$ and sends $\tau \in \mathrm{Cone}(B_j)$ to $\tau' \in (V(i,l) \setminus V(i-1,l)) \cap L_{\rho,\sigma}$ such that $\tau \subseteq_f \tau'$ and $\tau'(i) = j$ for each $j \in [p]$. $\square$

Corollary 5.18:

For $p \in \mathbb{N}$ and $n, l \in \mathbb{N}$ with $n \geq l$ it follows that

$$\|L_p(n,l)\| \simeq \|S_p(l)\|.$$

Proof. We know from Corollary 5.16 that

$$\|L_p(n,l)\| \simeq \|\mathrm{Cone}(K_p(l,l), (L_\sigma)_{\sigma \in P([l+1,n+1],[p])})\| = \|K\|.$$

Now this chain of equivalence follows

$$\|K\| \underset{3.18}{\overset{5.17}{\simeq}} \|K_p(l,l)\| \overset{5.11}{\simeq} \|\mathrm{sd}(S_p(l))\| \overset{3.13}{\simeq} \|S_p(l)\|. \qquad \square$$

Lemma 5.19:

Let $n, l \in \mathbb{N}$ with $n \geq l$ and let $p \in \mathbb{P}$, then for each $0 \leq i < l-1$ it holds that

$$\tilde{H}_i(J_p(n,l); \mathbb{Z}_p) \cong 0.$$

Proof. Let $C := \{\tau \in K_p(n,n) \colon |\mathrm{dom}(\tau)| < n-l\}$. From [5, p. 2513, Remark] (C is isomorphic to $E_{p,n-l}$) it follows that the dimension of $\|C\|$ is equal to $n-l$. Let $E^n = D^n \setminus C$ and let $J_p(n;a) = K_p(n;a)$ for all $a \in (\mathbb{Z}_p \setminus \{0\})^{n+1}$. Now it follows that

$$\|J_p(n,l)\| \simeq \left\| \bigvee_{a \in (\mathbb{Z}_p \setminus \{0\})^{n+1}} J_p(n;a) \right\| \tag{4}$$

with the same reasoning as in the proof of Lemma 5.12. Now by 3.35 we get that $\tilde{H}_i(\|J_p(n;a)\|) = \tilde{H}_i(\|K_p(n;a)\| \setminus \|C\|) \cong \tilde{H}^{n-i-1}(\|C\|)$. We know that the dimension of $\|C\|$ is equal to $n-l$ and thus $\tilde{H}^{n-i-1}(\|C\|) \cong 0$ for all $0 \leq i < l-1$. It follows that $\tilde{H}_i(\|J_p(n;a)\|) \cong 0$ for $0 \leq i < l-1$. The conditions of Theorem 3.35 are trivially fulfilled since all spaces involved are geometric realizations of finite simplicial complexes. Now by Lemma 3.34 and Equation 4 the claim is proven. $\square$

Lemma 5.20 (Sabok, [13, Lemma 16])**:**

Let $n, l \in \mathbb{N}$ with $n \geq l$ and let $p \in \mathbb{P}$, then for each $0 \leq i \leq l-2$ it holds that

$$\tilde{H}^i(K_p(n,l); \mathbb{Z}_p) \cong 0.$$

Proof. In the following all homology and cohomogy groups have coefficient in $\mathbb{Z}_p$.

Claim: $K_p(n,l)$ is connected. Consider a $\tau \in V(K_p(n,l))$ which is not a total function. Then this node is contained in a chain with a total function $\tau' \in V(K_p(n,l))$ as its largest element with respect to $\subseteq_f$. Since $K_p(n,l)$ is a simplicial complex it follows that $\{\tau, \tau'\} \in K_p(n,l)$. Now let τ and τ' be a total functions in $V(K_p(n,l))$ such that

$$\begin{aligned} \tau|_D = \tau'|_D, \quad D = [n+1] \setminus \{i\}, \\ \tau(i-1) = \tau(i) \neq \tau(i+1), \\ \tau(i+1) = \tau'(i) \end{aligned}$$

with $1 < i < n$. Then there are chains $C_1 = \{\tau_0, \ldots, \tau'', \tau\}$ and $C_2 = \{\tau_0, \ldots, \tau'', \tau'\}$ in $K_p(n,l)$ with respect to $\subseteq_f$. This means that τ and τ' are connected via τ''. It follows that every total function $K_p(n,l)$ is connected to the total zero function by induction on i. Let τ be a total function. If $\tau(n) = 0$ then this follows immediately from the reasoning above. If not then let τ' be the total function with $\tau|_{[n]} = \tau'|_{[n]}$ and $\tau'(n) = 0$. Then these functions are connected via τ'' with $\tau''|_{[n]} = \tau'|_{[n]} = \tau|_{[n]}$. Now the argument above can be applied to τ' and thus the claim is proven. Now consider the following Mayer-Vietoris-Sequence (see Theorem 3.38)

$$\begin{aligned} \cdots \to\ & \tilde{H}_i(\|J_p(n,l)\| \cap \|L_p(n,l)\|) \\ \to\ & \tilde{H}_i(\|J_p(n,l)\|) \oplus \tilde{H}_i(\|L_p(n,l)\|) \\ \to\ & \tilde{H}_i(\|J_p(n,l)\| \cup \|L_p(n,l)\|) \to \cdots \to 0. \end{aligned}$$

Remember that $J_p(n,l) \cap L_p(n,l) = K_p(n,l)$. Now by Lemma 5.19 we know that $\tilde{H}_i(\|J_p(n,l)\|) \cong 0$ and $\tilde{H}_i(\|L_p(n,l)\|) \cong 0$ by Corollary 5.18 and by the fact that the realization of $S_p(n)$ is the wedge of $(p-1)^{n+1}$ n-dimensional spheres for $0 \leq i < l-1$. Thus the direct sum of these groups is trivial. Also it is clear that $\tilde{H}_i(\|J_p(n,l)\| \cup \|L_p(n,l)\|) \cong 0$ by a similiar argument as in 5.19 also using the triviality of the cohomology and then applying Alexanders duality. The exactness of the sequence yields that $\tilde{H}_i(\|K_p(n,l)\|) \cong 0$ for $0 \leq i \leq l-2$ and thus by

Corollary 3.37 it follows that

$$\tilde{H}^i(\|K_p(n,l)\|) \cong 0$$

as claimed. $\square$

Now we can prove Theorem 5.1.

Proof. Firstly by Lemma 5.20 we know that for $0 \leq i < l-1$ we have that

$$\tilde{H}^i(\|K_p(n,l)\|, \mathbb{Z}_p) \cong 0.$$

Now let $f \in C(\|K_p(n,l)\|, \mathbb{R}^d)$. Notice that $d < l$ by the assumptions on d. Since the simplicial complex is compact the geometric realization is contained in a ball around the origin with finite radius $B = B_r(0)$ for $r \in \mathbb{R}_{>0}$ the map f can be resticted to a map $f_B = f|_B$. It remains to be shown that the map $\tilde{f} = \tilde{H}^{l-1}(f_B)\colon \tilde{H}^{l-1}(\|K_p(n,l)\|) \to \tilde{H}^{l-1}(B)$ has zero image. Since the reduced cohomology groups of this ball are all trivial because the ball is contractible we get that $\tilde{f}$ is the zero map.

We can conclude that the set $A(f)$ has at least one element and thus the claim of the Theorem is proven. $\square$

6 Bound on the chromatic numbers

In this last section a bound on the chromatic numbers of the graph $\Gamma(\varepsilon, \mathcal{P}, \mu)$ is derived with the help of the main result of the last section.

In this whole section let X be a set, $\varepsilon \in \mathbb{R}_{\geq 0}$ and let μ be a diffuse submeasure on the a subalgebra $\mathcal{B}$ of $\mathfrak{P}(X)$.

Definition 6.1:

Define for each $\delta \in \mathbb{R}_{\geq 0}$ the minimal cardinality of $\mathcal{P} \in \Pi(\mathcal{B})$ for which it holds that

$$\forall P \in \mathcal{P} \colon \mu(P) < \delta$$

as $k_\mu(\delta)$. This is well-defined since μ is diffuse. Furthermore define the map $K_\mu^\varepsilon \colon \mathbb{N} \to \mathbb{N}$ as

$$K_\mu^\varepsilon(n) = k_\mu\left(\frac{\varepsilon}{4n}\right)$$

and define for each $n \in \mathbb{N}$ the element $\mathcal{Q}_\mu^\varepsilon(n) \in \Pi(\mathcal{B})$ as

$$\mathcal{Q}_n^\varepsilon := \{I_1^\varepsilon, \ldots, I_k^\varepsilon\}$$

where $k = K_\mu^\varepsilon(n)$ and $\mu(I_i) < \frac{\varepsilon}{4n}$ for each $i = 1, \ldots, k$. ♠

Remark 6.2:

The map K_μ^ε is increasing and has the lower bound

$$K_\mu^\varepsilon(n) \geq \frac{4n}{\varepsilon} \cdot \mu(X)$$

for each $n \in \mathbb{N}$.

Proof. Firstly prove the bound on the map. To this end consider the measurable partition $\mathcal{Q}_\mu^\varepsilon(n)$. By definition we know that $\mu(P) < \frac{\varepsilon}{4n}$ for each $Q \in \mathcal{Q}_\mu^\varepsilon(n)$. Let $k = |\mathcal{Q}_\mu^\varepsilon(n)|$. It follows that

$$\mu(X) = \mu\left(\bigcup_{i=1}^k I_k\right) \leq \sum_{i=1}^k \mu(I_k) < \sum_{i=1}^k \frac{\varepsilon}{4n} = K_\mu^\varepsilon(n) \cdot \frac{\varepsilon}{4n}$$

and hence $K_\mu^\varepsilon(n) > \frac{4n}{\varepsilon} \cdot \mu(X)$. The fact that the map is increasing follows from this inequality. □

Assertion 6.3:

Let $F_\mu^\varepsilon \colon \mathbb{N} \to \mathbb{N}$ be an arbitrary increasing map such that

$$K_\mu^\varepsilon \circ F_\mu^\varepsilon = \mathrm{id}_{\mathbb{N}}.$$

The existence of such a map follows from the fact that K_μ^ε is increasing and unbounded by the previous remark.

Lemma 6.4:

The map F_μ^ε is unbounded.

Proof. Assume that there exists a bound on F_μ^ε. Since this map takes values in $\mathbb{N}$ and is increasing this means that the map eventually becomes constant. But since K_μ^ε is increasing and unbounded this would be a contradiction to the definition of F_μ^ε. □

Definition 6.5:

Define C_μ^ε as

$$C_\mu^\varepsilon := \sqrt[3]{\frac{\mu(X)^2}{16\varepsilon}}.$$

Additionally define $\mathcal{P}_\mu^\varepsilon(n) \in \Pi(\mathcal{B})$ with cardinality $N_\mu^\varepsilon(n)$ such that $\mu(P) < \frac{1}{n}$ for each $P \in \mathcal{P}_\mu^\varepsilon(n)$ and

$$\mathcal{Q}_\mu^\varepsilon(n) \preccurlyeq \mathcal{P}_\mu^\varepsilon(n)$$

for $n \leq F_\mu^\varepsilon(C_\mu^\varepsilon \cdot \sqrt[3]{n})$. ♠

The following Lemma is just an inequality needed to prove the main Theorem of this last section.

Lemma 6.6:

Let $d, n \in \mathbb{N}^+$ with $d < F_\mu^\varepsilon(C_\mu^\varepsilon \cdot \sqrt[3]{n})$ and let $k = K_\mu^\varepsilon(d)$. Then there is a $p \in \mathbb{P}$ dependent of the choice of k and d such that

$$\frac{dp+1}{n} < \frac{\varepsilon}{8}.$$

Proof. By Betrand's postulate[7] there exists a prime $p \in \mathbb{P}$ such that

$$k(d+1) < p < 2k(d+1).$$

Now assume that the claim of the Lemma is false. This would imply $dp \geq \frac{n\varepsilon}{8}$. By Remark 6.2 we know that

$$k \geq \frac{4d}{\varepsilon} \cdot \mu(X)$$

and thus

$$d+1 \leq 4d = \frac{\varepsilon}{\mu(X)} \left(\frac{4d}{\varepsilon} \mu(X) \right) \leq \frac{\varepsilon}{\mu(X)} k.$$

Now we can deduce that

$$\begin{aligned}
&\frac{n\varepsilon}{8} \leq dp \leq 2k(d+1)d < 2k(d+1)^2 \\
\Rightarrow &\frac{n\varepsilon}{16} < k(d+1)^2 \overset{*}{\leq} \frac{\varepsilon^2}{(\mu(X))^2} k^3 \\
\Rightarrow &K_\mu^\varepsilon(d) = k \overset{**}{>} C_\mu^\varepsilon \cdot \sqrt[3]{n}.
\end{aligned}$$

The inequality $*$ follows from the fact that $\frac{\varepsilon}{\mu(X)} k \geq d+1$ and thus the quantity is greater or equal to one. And the inequality $**$ follows by rearranging the inequality $*$ and the definition of C_μ^ε.

Since F_μ^ε is increasing by definition we can apply it to both sides yielding us

$$F_\mu^\varepsilon(K_\mu^\varepsilon(d)) = d \geq F_\mu^\varepsilon(C_\mu^\varepsilon \cdot \sqrt[3]{n})$$

contradicting the assumptions on d. ↯ □

Theorem 6.7 (Sabok, [13, Theorem 18])**:**
For each $\varepsilon \in \mathbb{R}_{>0}$ it holds that

$$\chi(\Gamma(\varepsilon, \mathcal{P}_\mu^\varepsilon(n), \mu)) \geq F_\mu^\varepsilon(C_\mu^\varepsilon \cdot \sqrt[3]{n}).$$

Proof. Let $n \in \mathbb{N}$ and define $k_n := N_\mu^\varepsilon(n) - 1$. Define the submeasure μ_n on $[k_n+1]$

[7] This is a Theorem from number theory proving the existence of a prime p for any $n \in \mathbb{N}$ with $n > 1$ such that $n < p < 2n$. For a proof of this see [3].

as

$$\mu_n(A) := \mu\left(\bigcup_{i\in A} A_i\right),\ \ A \in \mathfrak{P}([k_n+1])$$

where $\{A_0, \dots, A_{k_n}\} = \mathcal{P}_\mu^\varepsilon(n)$.

Now suppose that $d < F_\mu^\varepsilon(C_\mu^\varepsilon \cdot \sqrt[3]{n})$ and that there exists a coloring $c\colon \mathbb{Z}^{k_n+1} \to \{1, \dots, d\}$ of $\Gamma(\varepsilon, \mathcal{P}_\mu^\varepsilon(n), \mu)$. Let $k = K_\mu^\varepsilon(d)$.

Let $\pi \in \mathrm{Sym}([k_n+1])$ such that for $\mathcal{I} := \{I_1, \dots, I_k\} \in \Pi(\pi([k_n+1]))$ where the I_i are consecutive intervals[8] and $\mathcal{Q} := \{\bigcup_{i\in I} A_i \colon I \in \mathcal{I}\}$ it holds that

$$\forall Q \in \mathcal{Q}\colon \mu(Q) < \frac{1}{4d}.$$

This is possible because $\mathcal{Q} \preccurlyeq \mathcal{P}_\mu^\varepsilon(n)$ and because of the assumption on d.

Now let $p \in \mathbb{P}$ as described in the proof of Lemma 6.6 and let $l = dp$. For $f \in V_p(k_n, l)$ define the extension of f to a total function as

$$\bar{f}\colon [k_n+1] \to [p],\ \begin{cases} f(x),\ x \in \mathrm{dom}(f), \\ 0,\ \text{otherwise.} \end{cases}$$

We can extend the coloring c to $\bar{c}\colon V_p(k_n, l) \to \{1, \dots, d\}$ by $\bar{c}(f) = c(\bar{f})$ for each $f \in V_p(k_n, l)$. Now define the map $\tilde{c}$ as

$$\tilde{c}\colon V_p(k_n, l) \to \mathbb{R}^d,\ f \mapsto e_{\bar{c}(f)}.$$

This map can be affinely extended to a map $\|\tilde{c}\|$ on the geometric realization $\|K_p(k_n, l)\|$. Especially, this is a continuous map. Now since $l > d(p-1)$ we can apply Theorem 5.1 and get that there exists $x_0 \in \|K_p(k_n, l)\|$ such that the $\mathbb{Z}_p$-Orbit[9] of this point gets send to a single point by $\|\tilde{c}\|$.

We know that x_0 is contained in the image of a maximal chain $\{h_l, \dots h_0\}$ where $h_l \subseteq_f \dots \subseteq_f h_0$. Now choose $\mathbb{N} \ni i_0 < d$ such that the $(x_0)_{i_0} \neq 0$ (in the case that x_0 is the zero vector just choose another geometric realization which is isometric to the original one and just translated by some small vector). We can conclude

[8] This means that they contain a range of numbers $\{i, i+1, \dots, j-1, j\}$ for $0 \le i \le j \le k_n$ and that for each $i \in [k_n+1] \setminus \{0\}$ it holds that $\forall k \in I_i \forall j \in I_{i+1}\colon k < j$.

[9] Define the $\mathbb{Z}_p$ action on $\|K_p(k_n, l)\|$ as the affine extension of the $\mathbb{Z}_p$-action on $V_p(k_n, l)$ (see 5.5).

that there exists $m \in \{0, \ldots, l\}$ such that $\tilde{c}(h_m) = e_{i_0}$ and thus $\bar{c}(h_m) = i_0$. This follows from the fact that $\|\tilde{c}\|$ is an affine map which means that it preserves the barycentric coordinates of x_0. Also observe that for each $q \in \mathbb{Z}_p$ there is an $m_q \in \{0, \ldots, l\}$ such that $\bar{c}(h_{m_q} +_f q) = i_0$ since $\|\tilde{c}\|(x_0 +_f q) = \|\tilde{c}\|(x)$ (we can assume that $(x_0 +_f q)_{i_0} \neq 0$ by the same isometry argument as for the case where $q = 0$ since there are only finitely many q). Let $h = h_m$ and define the set A_h as

$$\begin{aligned} A_h := \{j \in \mathbb{Z}_p \colon \exists J \in \mathbb{I}(h)\colon& \\ &(h(J) = \{j\}, \exists I_1, I_2 \in \mathcal{I}\colon (I_1 \neq I_2 \ \wedge\ J \cap I_1 \neq \emptyset \neq J \cap I_2))\}. \end{aligned}$$

It follows that $|A_h| \leq k - 1$ since the elements of $\mathcal{I}$ are consecutive and disjoint and $|\mathcal{I}| = k$. Furthermore define B_h as

$$B_h := [p] \setminus A_h.$$

It follows immediately that $|B_h| \geq p - k + 1$. By definition of A_h it follows for each $j \in B_h$ that

$$h^{-1}(\{j\}) = \left\{ \bigcup_{m=1}^{M} J_m \colon \mathbb{N} \ni M \leq l + 1, \exists I \in \mathcal{I}\colon J_m \subseteq I \right\}.$$

By the choice of p we get

$$\begin{aligned} &\frac{k}{p} < \frac{1}{1+d} = 1 - \frac{d}{d+1} \\ \Rightarrow &1 - \frac{k}{p} > \frac{d}{d+1} \\ \Rightarrow &\left(1 - \frac{k}{p}\right)(d+1) > d \end{aligned}$$

and hence $(p - k)(d + 1) > dp = l$.

From $(p - k + 1)(d + 1) > l + 1$ and $|\mathbb{I}(h)| \leq l + 1$ we get the existence of an element $q_0 \in B_h$ such that $h^{-1}(\{q_0\})$ contains less than $d + 1$ component intervals of h. It follows by definition of $\mathcal{I}$ that

$$\mu_n(h^{-1}(\{q_0\})) \leq \frac{d\varepsilon}{4d} = \frac{\varepsilon}{4}. \tag{5}$$

Define $f := h_{m_{p-1-q_0}} +_f (p-1-q_0)$ and $g := h_{m_{p-q_0}} +_f (p-q_0)$. It remains to show that

$$\mu_n(\{i \in [k_n+1] \colon \bar{f}(i)+1 \neq \bar{g}(i)\}) < \varepsilon \tag{6}$$

which implies that $\bar{f}$ and $\bar{g}$ are connected in $\Gamma(\varepsilon, \mathcal{P}^{\varepsilon}_{\mu}(n), \mu)$. This will be a contradiction because

$$c(\bar{f}) = \bar{c}(f) = \bar{c}(h_{m_{p-1-q_0}} +_f (p-1-q_0)) = i_0 = \bar{c}(h_{m_{p-q_0}} +_f (p-q_0)) = \bar{c}(g) = c(\bar{g})$$

as we have established earlier. The functions h and $h_{m_q-1-q_0}$ differ in at most $l+1$ elements of $[k_n+1]$ since $h \subseteq_f h_{m_q-1-q_0}$ or $h_{m_q-1-q_0} \subseteq_f h$ and the fact that these functions have at most $l+1$ component intervals. The same reasoning works for $h_{m_q-q_0}$.

From Lemma 6.6 it follows that

$$\mu_n(\{i \in [k_n+1] \colon h_{m_q-1-q_0}(i) \neq h(i)\}) < \frac{\varepsilon}{8}, \tag{7}$$

$$\mu_n(\{i \in [k_n+1] \colon h_{m_q-q_0}(i) \neq h(i)\}) < \frac{\varepsilon}{8}. \tag{8}$$

On the one hand we can deduce that

$$\begin{aligned}
&\mu_n(\{i \in [k_n+1] \colon \bar{f} +_p 1 \neq \bar{f}+1\}) = \mu_n(\bar{f}^{-1}(\{p-1\})) \\
\overset{*}{\leq}\ & \mu_n(f^{-1}(\{p-1\})) + \frac{\varepsilon}{8} \\
\overset{\text{Def.f}}{\leq}\ & \mu_n(h^{-1}_{m_q-1-q_0}(\{q_0\})) + \frac{\varepsilon}{8} \\
\overset{(7)}{\leq}\ & \mu_n(h^{-1}(\{q_0\})) + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \\
\overset{(5)}{\leq}\ & \frac{\varepsilon}{8} + \frac{\varepsilon}{4}.
\end{aligned}$$

The inequality $*$ follows from $f \subseteq_f \bar{f}$ and $|\mathrm{dom}(\bar{f})| - |\mathrm{dom}(f)| \leq l$. Furthermore notice that

$$\begin{aligned}
&\mu_n(\{i \in [k_n+1] \colon \bar{f}(i) +_p 1 \neq h(i) +_p (p-q_0)\}) \\
=\ & \mu_n(\{i \in [k_n+1] \colon \bar{f}(i) \neq h(i) +_p (p-1-q_0)\}) \\
\overset{*}{\leq}\ & \mu_n(\{i \in [k_n+1] \colon f(i) \neq h(i) +_p (p-1-q_0)\}) + \frac{\varepsilon}{8} \\
=\ & \mu_n(\{i \in [k_n+1] \colon h_{m_q-1-q_0}(i) +_q (p-1-q_0) \neq h(i) +_p (p-1-q_0)\}) + \frac{\varepsilon}{8}
\end{aligned}$$

$$= \mu_n(\{i \in [k_n+1] \colon h_{m_q-1-q_0}(i) \neq h(i)\}) + \frac{\varepsilon}{8}$$
$$\overset{(7)}{\leq} \frac{\varepsilon}{4}.$$

The inequality $*$ follows from the same reasoning like above. Additionally we get that

$$\begin{aligned}
&\mu_n(\{i \in [k_n+1] \colon \bar{g}(i) \neq h(i) +_p (p-q_0)\}) \\
\overset{*}{\leq}\ &\mu_n(\{i \in [k_n+1] \colon g(i) \neq h(i) +_p (p-q_0)\}) + \frac{\varepsilon}{8} \\
=\ &\mu_n(\{i \in [k_n+1] \colon h_{m_q-q_0}(i) +_p (p-q_0) \neq h(i) +_p (p-q_0)\}) + \frac{\varepsilon}{8} \\
=\ &\mu_n(\{i \in [k_n+1] \colon h_{m_q-q_0}(i) \neq h(i)\}) + \frac{\varepsilon}{8} \\
\overset{(8)}{\leq}\ &\frac{\varepsilon}{4}.
\end{aligned}$$

Again the inequality $*$ follows like above.

From the last three inequalities the claim in (6) follows as needed. □

Theorem 4.4 and this Theorem now imply the following result.

Theorem 6.8 (Sabok, [13, Theorem 1])**:**
For any non-trivial, Hausdorff, abelian topological group G and arbitrary diffuse submeasure μ the group $L_0(\mu, G)$ is extremely amenable.

Proof. This follows directly from Theorem 4.4 and Theorem 6.7. □

A Submeasures

Definition A.1:

Let $\mathcal{A}$ be a Boolean algebra and $\mu\colon \mathcal{A} \to [0, \infty)$. The map μ is called a **submeasure** if

i.) $\mu(0) = 0$,

ii.) $\forall A, B \in \mathcal{A}\colon A \leq B \Rightarrow \mu(A) \leq \mu(B)$,

iii.) $\forall A, B \in \mathcal{A}\colon \mu(A \vee B) \leq \mu(A) + \mu(B)$. ♠

The thesis will assume that every submeasure is a submeasure on the a subalgebra of the power set algebra of a set X. This is possible because of the representation Theorem of stone [16].

Definition A.2:

Let X be a set. Then $\Pi(X)$ is the set of all finite partitions of X, which means:

$$\Pi(X) := \left\{\mathcal{A} \subseteq \mathfrak{P}_{\text{fin}}(X)\colon (\forall A, B \in \mathcal{A}\colon A \cap B = \emptyset) \wedge \bigcup \mathcal{A} = X\right\}.$$

Let $\mathcal{P}_1, \mathcal{P}_2 \in \Pi(X)$. $\mathcal{P}_1$ is said to **refine** $\mathcal{P}_2$ ($\mathcal{P}_2 \preccurlyeq \mathcal{P}_1$) if

$$\forall A \in \mathcal{P}_1 \exists B \in \mathcal{P}_2\colon A \subseteq B.$$

For every $\mathcal{P} \in \Pi(X)$ define the map $\iota_{\mathcal{P}}\colon X \to \mathcal{P}$ with

$$\iota_{\mathcal{P}}\colon x \mapsto \begin{cases} A_1, \text{ if } x \in A_1, \\ A_2, \text{ if } x \in A_2, \\ \vdots \\ A_n, \text{ if } x \in A_n, \end{cases}$$

where $k = |\mathcal{P}|$. The set $\Pi(\mathcal{B})$ contains all finite partitions of X but with the additional constraint that they have to be measurable with respect to a submeasure μ on the Boolean algebra of subsets $\mathcal{B}$ of X.

Let $f\colon X \to Y$ be a step function for a set Y. Then f is **measurable** if

$$\forall P \in \mathcal{P}_f\colon f^{-1}(P) \in \mathcal{B}.$$ ♠

Definition A.3:

Let X be a set and let μ be a submeasure on a subalgebra $\mathcal{B}$ of $\mathfrak{P}(X)$. Then μ is called **diffuse** if

$$\forall \varepsilon \in \mathbb{R}_{>0} \exists \mathcal{P} \in \Pi(\mathcal{B}) \forall P \in \mathcal{P} : \mu(P) < \varepsilon. \qquad \spadesuit$$

Definition A.4:

Let $(P, \leq)$ be a partially ordered set. The set is called directed if for every $p_1, p_2 \in P$ there exists an element $p_3 \in P$ such that

$$p_1 \leq p_3 \ \wedge \ p_2 \leq p_3. \qquad \spadesuit$$

Theorem A.5:

Let X be a set. Then the set $\Pi(X)$ together with the binary operation $\preccurlyeq$ is a directed set.

Proof. Reflexivity and transitivity follow directly from the reflexivity and transitivity of the $\subseteq$ relation. Now let $\mathcal{P}_1, \mathcal{P}_2 \in \Pi(X)$. Define $\mathcal{P}_3 := \{P \cap P' \colon P \in \mathcal{P}_1, P' \in \mathcal{P}_2\}$. Since for any two sets A, B it holds that $A \cap B \subseteq A$ and $A \cap B \subseteq B$ it follows that

$$\mathcal{P}_1 \preccurlyeq \mathcal{P}_3 \wedge \mathcal{P}_2 \preccurlyeq \mathcal{P}_3. \qquad \square$$

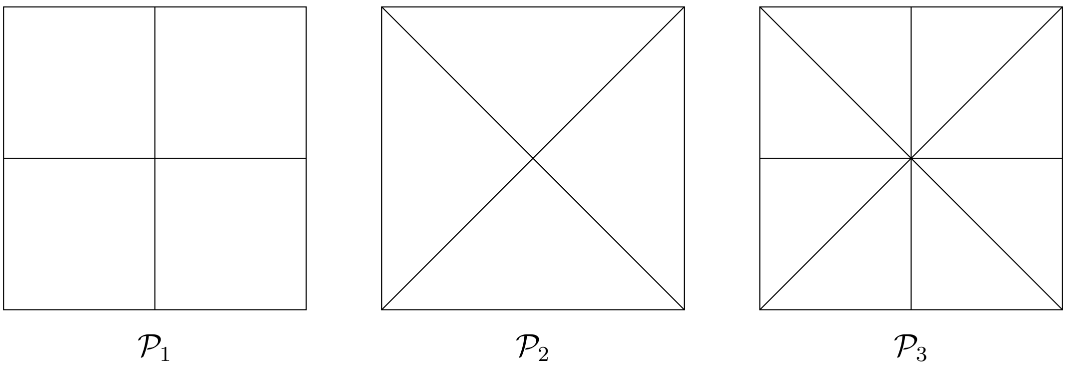

Figure 6: Example of 2 partions of the square $\mathcal{P}_1$ and $\mathcal{P}_2$ with common upper bound $\mathcal{P}_3$.

Remark A.6:

If X is an infinite set then $\Pi(X)$ is also infinite.

Proof. Define $\mathcal{P} \subseteq \Pi(X)$ as follows

$$\mathcal{P} := \Big\{\{\{x\},\ X \setminus \{x\}\} \colon\ x \in X\Big\}.$$

It is now clear that $|\mathcal{P}| = |X|$ and since $\mathcal{P} \subseteq \Pi(X)$ it follows that

$$|X| \leq |\Pi(X)|\,. \qquad \square$$

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